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Nonlinear Stability in Resonant Cases: A Geometrical Approach

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Summary. In systems with two degrees of freedom, Arnold's theorem is used for studying nonlinear stability of the origin when the quadratic part of the Hamiltonian is a nondefinite form. In that case, a previous normalization of the higher orders is needed, which reduces the Hamiltonian to homogeneous polynomials in the actions. However, in the case of resonances, it could not be possible to bring the Hamiltonian to the normal form required by Arnold's theorem. In these cases, we determine the stability from analysis of the normalized phase flow. Normalization up to an arbitrary order by Lie-Deprit transformation is carried out using a generalization of the Lissajous variables.

Key words. nonlinear stability, normal forms

1. Introduction

Arnold's theorem [3] can be used for studying the nonlinear stability in two-degrees-of-freedom Hamiltonian systems when the quadratic part of the Hamiltonian (corresponding to the expansion of the Hamiltonian in a neighborhood of the equilibria) is not sign-defined and, hence, Liapunov's stability theorem [20] cannot be applied. In that case, the linear motion in the neighborhood of the origin is equivalent to the subtraction of two harmonic oscillators with frequencies $\omega_1, \omega_2 > 0$. Under the assumption of no resonances among the frequencies ω_1 and ω_2, the Hamiltonian can be reduced to the Birkhoff normal form in action-angle variables $(I_1, I_2, \phi_1, \phi_2)$,

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4 + \cdots + \mathcal{H}_{2n} + \tilde{\mathcal{H}}, \tag{1}$$

where $\mathcal{H}_2 = \omega_1 I_1 - \omega_2 I_2$ and $\mathcal{H}_{2k} = \mathcal{H}_{2k}(I_1, I_2)$ are homogeneous polynomials of degree k in I_1, I_2 and $\tilde{\mathcal{H}}$ has a series expansion in the variables with terms at least of order $2n + 1$.

According to the formulation of Arnold's theorem given by Meyer and Schmidt [19], the origin of the system of differential equations derived from (1) is stable if for some k, with $2 \leq k \leq n$, the discriminant $D_{2k} = \mathcal{H}_{2k}(\omega_2, \omega_1)$ is not null, or equivalently if $\mathcal{H}_2$ does not divide $\mathcal{H}_{2k}$.

As Meyer and Schmidt [19] acknowledge, there are several implicit assumptions in stating that $\mathcal{H}$ is of the above form. Indeed, since $\mathcal{H}_2, \ldots, \mathcal{H}_{2n}$ are homogeneous polynomials in the actions I_1 and I_2, $\mathcal{H}$ is in the Birkhoff's normal form of degree $2n$ and, hence, some nonresonance assumption on the frequencies ω_i is implicit. This assumption is what Deprit and Deprit-Bartholomé [7] named the *general condition of irrationality*, which consists of assuming that for any pair (k_1, k_2) of rational integers, $k_1\omega_1 + k_2\omega_2 \neq 0$. In these conditions, this theorem has been proved very useful to determine the orbital stability of equilibria in several problems like the restricted three-body problem [7], [19], [23] or the problem of geostationary satellites [10], [16], to mention but a few.

For the resonant cases, Arnold's theorem [3] is inapplicable and the stability problem requires a special treatment. Several attempts have been made to solve the stability problem in resonant cases (e.g. Markeev [17], [18]; Sokolsky [24], [25], [26]), where specific resonances are treated. Recently, Cabral and Meyer [6] revisited the problem and gave a general result that applies for both nonresonant and resonant cases.

In this paper, we propose to determine the nonlinear stability from a geometrical point of view—more precisely, by analyzing the normalized phase flow around the equilibrium. Our procedure is valid for whatever resonance. Once the system has been normalized up to the required order (for instance, by a Lie-Deprit transformation [8]), we express the Hamiltonian in the so-called invariants [13], [21] and look at the geometry of the orbits on the manifold where the origin lies in the reduced phase space. Closed orbits around the origin imply stability, whereas outgoing asymptotic trajectories ensure instability. Moreover, if the origin is unstable, we find unstable critical points in a neighborhood of the manifold where the origin lies.

The difficulty of the procedure could lie in the analysis of the normalized phase flow, but the method can be used for any kind of resonance that prevent application of Arnold's theorem. The normalization is carried out in the extended Lissajous variables [11], especially designed to handle resonant harmonic oscillators. Since the Lie derivative in this set of canonical variables is very simple (Sect. 2), normalization is also very easy to perform, even at high orders. Besides, it is also known [12] that the topology of the reduced space phase is a surface of revolution; hence it is not difficult to represent graphically the phase flow on such surface and to recognize the type of trajectories.

As an illustration, we consider (Sect. 3) the resonant cases of the Lagrangian points in the restricted three-body problem. To this problem, Deprit and Deprit-Bartholomé [7] and Meyer and Schmidt [19] applied Arnold's theorem to determine the stability of the Lagrangian point for every value of the mass ratio μ except for the resonant cases 2:1 and 3:1. These cases where analyzed by a different method by Markeev [17] and Alfriend [1], [2], who proved that for these resonances the equilateral point was unstable, indeed.

2. Extended Lissajous Transformation

Let us assume we have a Hamiltonian of the type

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{P} = \mathcal{H}_2 + \sum_{n>0} \epsilon^n \mathcal{H}_{n+2}(x_1, x_2, X_1, X_2),$$

where

$$\mathcal{H}_2 = \tfrac{1}{2}(X_1^2 + \omega_1^2 x_1^2) - \tfrac{1}{2}(X_2^2 + \omega_2^2 x_2^2),$$

$\mathcal{H}_{n+2}$ is a homogeneous polynomial of degree $n+2$ in the Cartesian variables (x_1, x_2, X_1, X_2), and ϵ is a small parameter.

These systems are called *semisimple* [5] because their dominant term leads to a linear Hamiltonian vector field that is semisimple. The concept of normalization for semisimple systems in equilibrium at the origin must be credited to Whittaker [28] who applied Poincaré's *nouvelle méthode* [22]; Birkhoff [4] provides a different method for the normalization (for an automated development of the generating function, see e.g. [14]). The introduction of the Lie transformation in the 1960s [15], [8] made easier the automatization of the normalization. When the normalization is carried out by a Lie method, especially when high orders are required, one must be aware of the fact that, the simpler the Lie derivative associated with the unperturbed Hamiltonian, the easier the application of the method is. In this regards, the extended Lissajous variables [11] are of great utility for normalizing this type of semisimple Hamiltonians.

Let us assume, too, that the frequencies are in resonance $p\colon q$; that is to say, there are two coprime integers p and q and a frequency ω such that $\omega_1 = p\omega$ and $\omega_2 = q\omega$.

Under these hypotheses, the extended Lissajous transformation [11]

$$\begin{array}{rcc} f\colon\ T^2 \times \{\Psi_1 > 0\} \times \{|\Psi_2| \le \Psi_1)\} & \longmapsto & \mathbb{R}^4, \\ (\psi_1, \psi_2, \Psi_1, \Psi_2) & \longmapsto & (x_1, x_2, X_1, X_2), \end{array}$$

is defined by

$$\begin{aligned} x_1 &= \sqrt{\frac{\Psi_1 + \Psi_2}{\omega_1 p}} \sin p(\psi_1 + \psi_2) &&= s/p \sin p(\psi_1 + \psi_2), \\ x_2 &= \sqrt{\frac{\Psi_1 - \Psi_2}{\omega_2 q}} \sin q(\psi_1 - \psi_2) &&= d/q \sin q(\psi_1 - \psi_2), \\ X_1 &= \sqrt{\frac{\omega_1(\Psi_1 + \Psi_2)}{p}} \cos p(\psi_1 + \psi_2) &&= \omega s \cos p(\psi_1 + \psi_2), \\ X_2 &= \sqrt{\frac{\omega_2(\Psi_1 - \Psi_2)}{q}} \cos q(\psi_1 - \psi_2) &&= \omega d \cos q(\psi_1 - \psi_2), \end{aligned} \tag{2}$$

where we used the shorthand $s^2 = (\Psi_1 + \Psi_2)/\omega$ and $d^2 = (\Psi_1 - \Psi_2)/\omega$ to remove the irrational expressions from the definition.

The pullback of the Hamiltonian $\mathcal{H}_2$ is

$$f^{\#}\mathcal{H}_2 = \omega \Psi_2, \tag{3}$$

and $f^{\#}\mathcal{P}$ is a Fourier series in the angles ψ_1, ψ_2, with coefficients in the real algebra of homogeneous polynomials of degree n (with $n > 2$) in s and d.

The main advantage of this set of canonical coordinates is that the Lie derivative associated with the Hamiltonian (3) is very simple; namely, it is the operator

$$L_0(-) = \omega \frac{\partial(-)}{\partial \psi_2}. \tag{4}$$

Hence, when the perturbation is periodic in the Lissajous variable ψ_2, normalizing a perturbed elliptic oscillator amounts to averaging the dynamical system over this variable.

One critical issue that must be faced before undertaking a normalization is to prefigure the topological structure subjacent to the orbital space after reduction. This can be achieved in terms of some generators. Indeed, after normalization, Ψ_2 becomes a new integral, and this can be understood as a symmetry induced by Ψ_2 underlying a symmetry group. The invariants of this symmetry group are precisely the generators of the orbital space. They are functions belonging to the kernel of the Lie operator (4). In [12] the following set of functions is defined in terms of the Lissajous variables:

$$\begin{aligned} M_1(p,q) &= \tfrac{1}{2}\Psi_1, \\ M_2(p,q) &= \tfrac{1}{2}\Psi_2, \\ C_1(p,q) &= 2^{-(p+q)/2}(\Psi_1 - \Psi_2)^{p/2}(\Psi_1 + \Psi_2)^{q/2}\cos 2pq\psi_1, \\ S_1(p,q) &= 2^{-(p+q)/2}(\Psi_1 - \Psi_2)^{p/2}(\Psi_1 + \Psi_2)^{q/2}\sin 2pq\psi_1, \end{aligned} \tag{5}$$

belonging to the kernel of the Lie operator L_0. Obviously, they are not independent, but satisfy the relation

$$C_1^2 + S_1^2 = (M_1 + M_2)^q (M_1 - M_2)^p.$$

That is to say, the reduced phase space is a surface of revolution for a given value of M_2.

Since in our case Ψ_2 is an integral, the functions M_1, C_1 and S_1 determine a transformation

$$(x, y, X, Y) \longmapsto (M_1, C_1, S_1)$$

mapping the no definite form

$$X^2 - Y^2 + \omega^2(p^2x^2 - q^2y^2) = 2\omega\Psi_2 = 4\omega M_2$$

in the phase space $(\omega p x, \omega q y, X, Y)$ onto the two-dimensional surface of revolution $\mathcal{F}(M_2)$ given by (see Fig. 1)

$$\mathcal{F}(M_2)\colon\ C_1^2 + S_1^2 = (M_1 + M_2)^q (M_1 - M_2)^p. \tag{6}$$

It is worth noting that the normalized Hamiltonian can be expressed in a unique way in terms of the invariants M_2, M_1, C_1, and S_1.

The equations of the motion in this set of variables are obtained immediately by application of the Liouville-Jacobi theorem:

$$\dot{C}_1 = \{C_1, \mathcal{H}'\}, \qquad \dot{S}_1 = \{S_1, \mathcal{H}'\}, \qquad \dot{M}_1 = \{M_1, \mathcal{H}'\}. \tag{7}$$

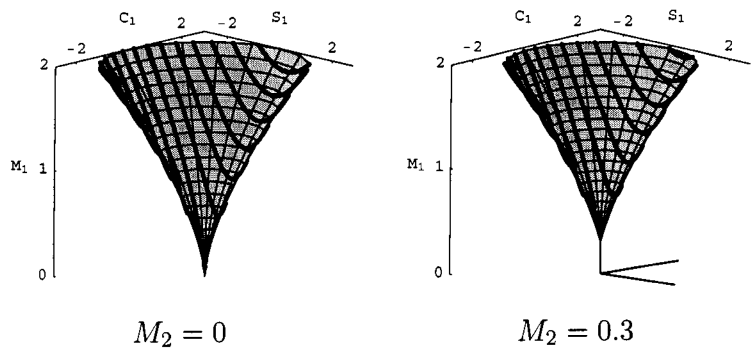

Fig. 1. Resonance 2:1. Phase flow of the reduced Hamiltonian on the manifold $M_2 = 0$ (left) and $M_2 = 0.3$ (right). There are two heteroclinic orbits passing through the origin, hence its instability.

By computing Poisson brackets, we can show that the Lie algebra is defined by

$$\begin{aligned}
&\{M_1; C_1\} = pqS_1,\\
&\{C_1; S_1\} = \tfrac{1}{2}pq(M_1 + M_2)^{q-1}(M_1 - M_2)^{p-1}((q - p)M_2 - (q + p)M_1),\\
&\{S_1; M_1\} = pqC_1.
\end{aligned} \tag{8}$$

N.B. For the resonance $p = q = 1$, we recover the second set of Lissajous variables given by Deprit [9], as well as the classical *rigid body* Poisson structure.

3. An Example: The Lagrangian Point

Let us consider the planar restricted three-body problem. In Cartesian variables $\boldsymbol{Z} = (Q_1, Q_2, P_1, P_2)$, its Hamiltonian may be written as

$$\mathcal{H} = \frac{1}{2}(P_1^2 + P_2^2) + P_1Q_2 - P_2Q_1 - \frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2}, \tag{9}$$

where ρ_1, ρ_2 are respectively the distances of the particle to the primaries of masses $1 - \mu$ and μ, placed at the points $(-\mu, 0)$ and $(1 - \mu, 0)$.

The Lagrangian equilibrium L_4 is the point $((1-2\mu)/2, \sqrt{3}/2)$. For details about this problem, the reader is addressed to any textbook on celestial mechanics (e.g. Szebehely [27]).

By a linear canonical transformation, we shift the origin of the coordinate system to L_4. Then, we expand the Hamiltonian in power series of the coordinates. Since the expansion is made in the neighborhood of an equilibrium, the constant part, i.e. the value of the Hamiltonian at the equilibrium, can be neglected and the linear part will be null. The expanded Hamiltonian can be arranged in the form

$$\mathcal{H} = \sum_{n\geq 2} \mathcal{H}n, \tag{10}$$

with $\mathcal{H}_n$ homogeneous polynomials of degree n in the variables. The terms up to the fifth order are

$$\mathcal{H}_2 = \frac{1}{2}(P_1^2 + P_2^2) - (P_2 Q_1 - P_1 Q_2) + \frac{1}{8}(Q_1^2 - 6\gamma Q_1 Q_2 - 5Q_2^2), \qquad (11)$$

$$\mathcal{H}_3 = \frac{\sqrt{3}}{48}(-7\gamma Q_1^3 + 9Q_1^2 Q_2 + 33\gamma Q_1 Q_2^2 + 9Q_2^3), \qquad (12)$$

$$\mathcal{H}_4 = \frac{1}{128}(37Q_1^4 + 100\gamma Q_1^3 Q_2 - 246 Q_1^2 Q_2^2 - 180\gamma Q_1 Q_2^3 - 3Q_2^4), \qquad (13)$$

where for the sake of simplicity we set $\gamma = \sqrt{3}(1 - 2\mu)$.

The system of linear differential equations derived from the quadratic term, $\mathcal{H}_2$, describes the tangent flow around L_4. Linear stability will be established according to the character of the associated eigenvalues, which are the roots of the characteristic equation

$$\lambda^4 + \lambda^2 + \frac{27}{16} - \frac{9}{16}\gamma^2 = 0.$$

If $\mu < \mu_c = (1 - \sqrt{23/27})/2$ (the so-called Routh's critical mass ratio), the four eigenvalues are distinct and purely imaginary and the equilibrium is linearly stable. In that case, the four eigenvalues are $\pm i\omega_1, \pm i\omega_2$, where the strictly positive numbers ω_1 and ω_2 are determined by unambiguously by the set of relations

$$0 < \omega_2 < 1/\sqrt{2} < \omega_1 < 1, \qquad \omega_1^2 + \omega_2^2 = 1, \qquad 16\omega_1^2\omega_2^2 = 27 - 9\gamma^2.$$

Once the linear stability has been established, the next step is to transform $\mathcal{H}_2$ into its normal form. We proceed in a similar way to [7] and build a linear canonical transformation to a new set of variables $\boldsymbol{z} = (q_1, q_2, p_1, p_2)$,

$$\boldsymbol{Z} = P\boldsymbol{z}, \qquad (14)$$

defined by the matrix

$$P = \frac{1}{2}\begin{pmatrix} 0 & 0 & l_1/k\omega_1 & -l_2^2/kl_2\omega_2 \\ -8\omega_1/kl_1 & -8\omega_2/kl_2 & -3\gamma/kl_1\omega_1 & 3\gamma/kl_2\omega_2 \\ -m_1\omega_1/kl_1 & -m_2\omega_2/kl_2 & 3\gamma/kl_1\omega_1 & -3\gamma/kl_2\omega_2 \\ 3\gamma\omega_1/kl_1 & 3\gamma\omega_2/kl_2 & n_1/kl_1\omega_1 & -n_2/kl_2\omega_2 \end{pmatrix},$$

where

$$k^2 = \omega_1^2 - \omega_2^2, \qquad \left.\begin{aligned} l_i^2 &= 9 + 4\omega_i^2 \\ m_i &= 1 + 4\omega_i^2 \\ n_i &= 9 - 4\omega_i^2 \end{aligned}\right\} \qquad (i = 1, 2).$$

In the new variables, the quadratic part (11) of the Hamiltonian is the normal form

$$\mathcal{H}_2 = \tfrac{1}{2}(p_1^2 + \omega_1^2 q_1^2) - \tfrac{1}{2}(p_2^2 + \omega_2^2 q_2^2),$$

and the terms (12) and (13) are homogeneous polynomials in the coordinates (q_1, q_2) of degree three and four, respectively, which are too long to be reproduced here.

At this point, both papers [7] and [19] proceed to the normalization of the Hamiltonian, assumed no resonances are met. Then, their study is valid except for the named mass ratios μ_2 and μ_3, values where the ratio of the frequencies ω_1 to ω_2 is 2:1 and 3:1, respectively. Here is where our method is applied.

3.1. Resonance 2:1

If $\mu = \mu_2 = \frac{1}{2}(1 - \sqrt{1833}/45)$, then $\omega_1 = 2/\sqrt{5}, \omega_2 = 1/\sqrt{5}$, and resonance 2:1 occurs. After applying transformation (14) to the Hamiltonian (10), we use the Lissajous transformation (2) with $\omega = 1/\sqrt{5}$, $p = 2, q = 1$. The pullback of Hamiltonians (11) and (12) are

$$\begin{aligned}\mathcal{H}_2 &= \tfrac{1}{\sqrt{5}}\Psi_2,\\ \mathcal{H}_3 &= H_{4,0} + H_{0,4} + H_{1,-1} + H_{2,2} + H_{3,-3} + H_{3,5} + H_{5,3} + H_{6,6},\end{aligned}$$

where $H_{i,j}$ stands for

$$H_{i,j} = c_{i,j}\cos(i\psi_1 + j\psi_2) + s_{i,j}\sin(i\psi_1 + j\psi_2), \tag{15}$$

with coefficients $c_{i,j}$ and $s_{i,j}$ depending on the momenta Ψ_1 and Ψ_2.

A first-order normalization is made simply by averaging over the angle ψ_2, which gives

$$\mathcal{H}_3' = 2^{-3/2}(\Psi_1 - \Psi_2)\sqrt{(\Psi_1 + \Psi_2)}(k_c\cos 4\psi_1 + k_s\sin 4\psi_1),$$

with

$$k_c = -5^{(1/4)}\frac{551}{882}\sqrt{\frac{611}{610}}, \qquad k_s = -5^{(1/4)}\frac{229}{147}\sqrt{\frac{1}{122}}.$$

Taking into account the functions defined in (5), the averaged Hamiltonian can be written as

$$\mathcal{H}_3' = k_c C_1 + k_s S_1. \tag{16}$$

For each manifold M_2 = constant the normalized phase space yields over the surface of revolution (6), that in this case ($p = 2, q = 1$) is

$$C_1^2 + S_1^2 = (M_1 + M_2)(M_1 - M_2)^2. \tag{17}$$

The trajectories are the level contours of the plane (16) over the surface (17).

Let us recall that we are interested in determining the stability of the origin, which is mapped to the vertex of the surface (17) when $M_2 = 0$. On this manifold, the trajectories are the intersection of the surface of revolution $C_1^2 + S_1^2 = M_1^2$, with planes, defined by (16), that are parallel to the M_1 axis. In particular, the plane $k_c C_1 + k_s S_1 = 0$ gives rise to two asymptotic orbits to the origin, one of them an outgoing trajectory, and thus the origin is unstable; the rest of the orbits are escape orbits.

For $M_2 \neq 0$, a similar portrait is obtained. Moreover, the vertex of each manifold (17), when $M_2 \neq 0$, is an unstable critical point, as can be derived from the linearized equations of the motion around it. Then, we can conclude that in resonant case 2:1, the Lagrangian point L_4 is an unstable equilibrium, which agrees with the result given by Alfriend [1].

3.2. Resonance 3:1

Resonance 3:1 happens at the mass ratio $\mu = \mu_3 = \frac{1}{2}(1 - 2\sqrt{117}/45)$, in which case $\omega_1 = 3/\sqrt{10}$, $\omega_2 = 1/\sqrt{10}$. In terms of the Lissajous variables (now $\omega = 1/\sqrt{10}$, $p = 3$, $q = 1$), the expressions of the Hamiltonians (11), (12), and (13) are

$$\begin{aligned}
\mathcal{H}_2 &= \tfrac{1}{\sqrt{10}}\Psi_2,\\
\mathcal{H}_3 &= H_{1,-1} + H_{3,-3} + H_{3,3} + H_{1,5} + H_{5,1} + H_{5,7} + H_{7,5} + H_{9,9},\\
\mathcal{H}_4 &= -\tfrac{29871}{8192}d^4 - \tfrac{3063}{2048}d^2s^2 - \tfrac{1021}{24576}s^4\\
&\quad + H_{2,-2} + H_{4,-4} + H_{0,6} + H_{2,4} + H_{4,2} + H_{6,0}\\
&\quad + H_{4,8} + H_{6,6} + H_{8,4} + H_{8,10} + H_{10,8} + H_{12,12}.
\end{aligned}$$

Inasmuch as every term in $\mathcal{H}_3$ depends on the angle ψ_2, it does not contribute to the first-order normalization, since its average over this angle vanishes. Hence, it is necessary to go one order further with the normalization. The resulting second-order normalized Hamiltonian is

$$\mathcal{H}_2' = \frac{1}{\sqrt{10}}\Psi_2, \qquad \mathcal{H}_3' = 0,$$

and

$$\mathcal{H}_4' = \alpha M_1^2 + \beta M_1 M_2 + \gamma M_2^2 + \kappa C_1 + \sigma S_1,$$

which has the coefficients

$$\begin{aligned}
\alpha &= -519/560, &\qquad \kappa &= \tfrac{500347}{5640\sqrt{329}},\\
\beta &= -389/420, & &\\
\gamma &= 3319/1680, &\qquad \sigma &= \tfrac{6479}{564}\sqrt{\tfrac{71}{3290}}.
\end{aligned}$$

The normalized phase space yields over the revolution surface (6) for each M_2 = constant. Recall that now $p = 3$ and $q = 1$; thus, this surface is

$$C_1^2 + S_1^2 = (M_1 + M_2)(M_1 - M_2)^3. \tag{18}$$

As was mentioned above, the equilibrium (that is, the point whose stability we are trying to determine) is the minimum of the surface (18) for $M_2 = 0$; thus, we focus on the kind of trajectories we find on the manifold

$$C_1^2 = S_1^2 = M_1^4,$$

and the trajectories will be the level contours of the function

$$\mathcal{H}_4' = \alpha M_1^2 + \kappa C_1 + \sigma S_1$$

over this manifold.

By virtue of the Lie-Poisson structure (8), the equations of motion are

$$\begin{aligned}
\dot{C}_1 &= -6(\alpha M_1 S_1 + \sigma M_1^3),\\
\dot{S}_1 &= 6(\alpha M_1 C_1 + \kappa M_1^3),\\
\dot{M}_1 &= 3(\kappa S_1 - \sigma C_1),
\end{aligned} \tag{19}$$

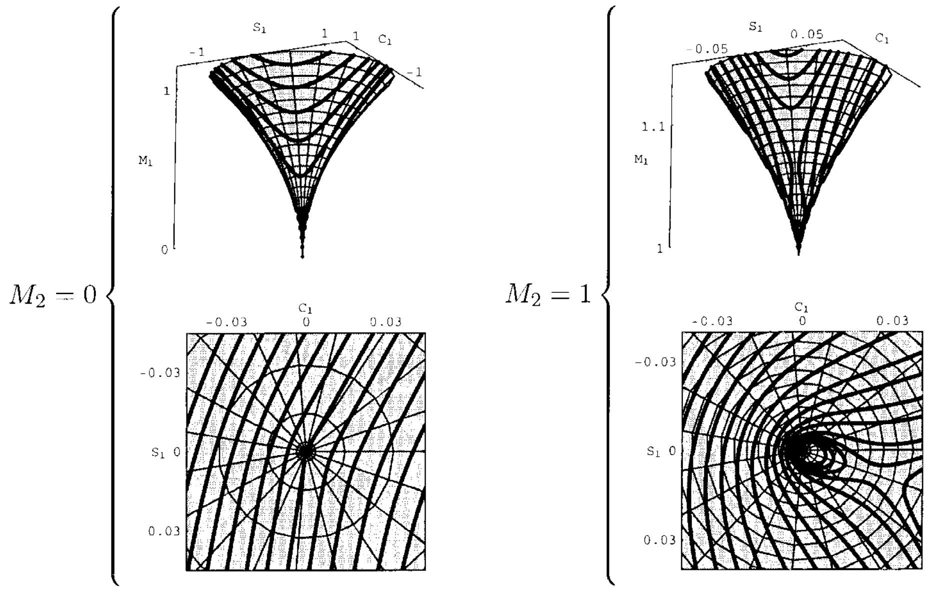

Fig. 2. Resonance 1:3. Phase flow of the reduced Hamiltonian. Above, a general view, and below, a view from the vertical axis. Left: Manifold $M_2 = 0$; there are two heteroclinic orbits passing through the origin, hence its instability. Right: Manifold $M_2 > 0$; there are two equilibria, one stable (the vertex of the surface) and the other unstable. The latter goes toward the former when $M_2 \to 0$ as the consequence of a saddle-node bifurcation.

and on the manifold $C_1^2 + S_1^2 = M_1^4 (M_2 = 0)$, there is only one equilibrium point, namely the origin $C_1 = S_1 = M_1 = 0$, that is unstable. Indeed, the curve

$$\left.\begin{array}{l} \alpha M_1^2 + \kappa C_1 + \sigma S_1 = 0, \\ C_1^2 + S_1^2 = M_1^4, \end{array}\right\}$$

determines two asymptotic orbits passing through the origin, one of them an outgoing trajectory (see Fig. 2, left). Thus, the point is unstable.

It is worth mentioning a notable difference with the preceding case. For the resonance 2:1, for whatever the value of M_2, all the orbits, except those attached to the critical points, are escape trajectories. However, for the 3:1 resonance, the phase flow for $M_2 \neq 0$ is quite different. Indeed, for a manifold $M_2 \neq 0$, there are two equilibria, one unstable, and the other the minimum of the surface (18), which is now stable. Nevertheless, as we shall see below, the phase portrait for $M_2 = 0$ is obtained by continuity through a saddle-node bifurcation.

Let us consider now the motion on a manifold $M_2 =$ constant $\neq 0$. On account of the Lie-Poisson structure (8), the equations of motion are

$$\begin{aligned} \dot{C}_1 &= -3S_1(2\alpha M_1 + \beta M_2) - 3\sigma (M_1 - M_2)^2 (2M_1 + M_2), \\ \dot{S}_1 &= 3C_1(2\alpha M_1 + \beta M_2) + 3\kappa (M_1 - M_2)^2 (2M_1 + M_2), \\ \dot{M}_1 &= 3(\kappa S_1 - \sigma C_1). \end{aligned} \tag{20}$$

The possible equilibria are in fact the local extrema of $\mathcal{H}_4'$ on the surface (18). Thus, in order to find them we put

$$\begin{aligned} C_1 &= \frac{-\kappa(M_1 - M_2)^2(2M_1 + M_2)}{2\alpha M_1 + \beta M_2}, \\ S_1 &= \frac{-\sigma(M_1 - M_2)^2(2M_1 + M_2)}{2\alpha M_1 + \beta M_2}, \end{aligned}$$

which are solutions of the system made by zeroing equations (20), into the equation of the surface (18). By doing so, M_1 is obtained as a root of the equation

$$\begin{aligned} \frac{(M_1 - M_2)^3}{(2\alpha M_1 + \beta M_2)^2} \big(& M_2^3(\beta^2 + \kappa^2 + \sigma^2) + M_1 M_2^2(4\alpha\beta + \beta^2 + 3\kappa^2 + 3\sigma^2) \\ & + M_1^2 M_2 4\alpha(\alpha + \beta) + 4M_1^3(\alpha^2 - \kappa^2 - \sigma^2)\big) = 0. \end{aligned}$$

Since by definition $M_1 \geq |M_2|$, there are two possible equilibria: one the point O of coordinates $C_1 = S_1 = 0$, $M_1 = M_2 \geq 0$, that is, the minimum of the surface, and the second one, P whose coordinate M_1 is a root of the cubic equation. Replacing the numerical values of the coefficients, only one real root results, namely $M_1 = 1.066271959 M_2$, that exists only when $M_2 > 0$. The equilibrium point corresponding to the cubic is

$$C_1 = 0.0231827 M_2^2, \qquad S_1 = 0.00799888 M_2^2, \qquad M_1 = 1.066271959 M_2.$$

In order to see the stability of these points, we compute the eigenvalues of the variational equations, that is, the system

$$\begin{aligned} \delta\dot{C}_1 &= -3(2\alpha M_1 + \beta M_2)\delta S_1 - 6(\alpha S_1 + 3M_1^2\sigma - 3M_1 M_2 \sigma)\delta M_1, \\ \delta\dot{S}_1 &= 3(2\alpha M_1 + \beta M_2)\delta C_1 + 6(\alpha C_1 + 3M_1^2\kappa - 3M_1 M_2 \kappa)\delta M_1, \\ \delta\dot{M}_1 &= -3\sigma\delta C_1 + 3\kappa\delta S_1. \end{aligned} \tag{21}$$

At the point $O = (0, 0, M_2)$, the characteristic equation of this linear system is

$$\lambda(\lambda^2 + (3(2\alpha + \beta)M_2)^2) = 0;$$

hence, this point is stable if $M_2 \neq 0$ and unstable for $M_2 = 0$.

With respect to the second point P, its characteristic equation is

$$\lambda(\lambda^2 - 24.2054103393 M_2^2) = 0,$$

which has one real positive root; consequently, the equilibrium, if any, is unstable (see Fig. 2, right).

N.B. The eigenvalue $\lambda = 0$ is a consequence of the existence of the Casimir (18).

Consequently, for $M_2 > 0$ there are two equilibria, one stable (O) and the other unstable (P); for $M_2 = 0$ there is only one (O) that is unstable; and there is none for $M_2 < 0$.

But the point P goes toward O as $M_2 \to 0$, until both coalesce for $M_2 = 0$. Thus, the origin O is unstable as a consequence of the saddle-node bifurcation that takes place (see Fig. 2, right).

Henceforth, we can conclude that for resonant case 3:1 the Lagrangian point is an unstable point; this result agrees with that given by other authors [18], [2].

4. Conclusions

The extended Lissajous variables is shown to be very useful for normalizing perturbed Hamiltonians with an unperturbed part made of harmonic oscillators in resonance. The phase portrait of the reduced Hamiltonian over the manifold $M_2 = 0$ determines the stability of the equilibria.

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