Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 323 (2006) 57–62
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

A transform involving Chebyshev polynomials and its inversion formula

Óscar Ciaurri [a,1], Luis M. Navas [b], Juan L. Varona [a,*,1]

[a] *Departamento de Matemáticas y Computación, Universidad de La Rioja, Calle Luis de Ulloa s/n, 26004 Logroño, Spain*

[b] *Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1-4, 37008 Salamanca, Spain*

Received 9 March 2005

Available online 10 November 2005

Submitted by B.C. Berndt

Abstract

We define a functional analytic transform involving the Chebyshev polynomials $T_n(x)$, with an inversion formula in which the Möbius function $\mu(n)$ appears. If $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, then given a bounded function from $[-1, 1]$ into $\mathbb{C}$, or from $\mathbb{C}$ into itself, the following inversion formula holds:

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{n^s} f\bigl(T_n(x)\bigr)$$

if and only if

$$f(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} g\bigl(T_n(x)\bigr).$$

Some other similar results are given.

Keywords: Chebyshev polynomials; Möbius function; Möbius transform; Dirichlet convolution; Inversion formula

* Corresponding author.

E-mail addresses: oscar.ciaurri@dmc.unirioja.es (Ó. Ciaurri), navas@usal.es (L.M. Navas), jvarona@dmc.unirioja.es (J.L. Varona).

URL: http://www.unirioja.es/dptos/dmc/jvarona/welcome.html (J.L. Varona).

[1] Research of the author supported by grant BFM2003-06335-C03-03 of the DGI.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.10.028

1. Introduction and main results

If we have an arithmetical function $\alpha:\mathbb{N}\to\mathbb{R}$ (or $\mathbb{C}$) and a function $f:(0,\infty)\to\mathbb{R}$ (or $\mathbb{C}$), we can define a new function $g=\alpha\circ f$ by taking

$$g(x)=(\alpha\circ f)(x)=\sum_{n=1}^{\infty}\alpha(n)f\left(\frac{x}{n}\right),\quad x\in(0,\infty). \tag{1}$$

Moreover, let us suppose that α is invertible with respect to Dirichlet convolution (this happens if and only if $\alpha(1)\neq 0$). Then, it is well known that we have the inversion formula $f=\alpha^{-1}\circ g$. A typical case is when α is a completely multiplicative function; in this case $\alpha^{-1}(n)=\mu(n)\alpha(n)$, where $\mu(n)$ is the Möbius function. Thus, we have

$$f(x)=\sum_{n=1}^{\infty}\mu(n)\alpha(n)g\left(\frac{x}{n}\right),\quad x\in(0,\infty)$$

(see, for instance, [1]). A common example of a completely multiplicative function is $\alpha(n)=n^{-s}$, $s\in\mathbb{C}$, which gives rise to Dirichlet series.

In this paper we present a new transform/inverse pair in which both the Chebyshev polynomials $\{T_n(x)\}_{n=1}^{\infty}$ and the Möbius function $\mu(n)$ appear. The Chebyshev polynomials satisfy many identities and orthogonal conditions, but for our purposes only the property

$$T_m\big(T_n(x)\big)=T_{mn}(x) \tag{2}$$

is essential. For $x\in[-1,1]$, this formula is clear from $T_k(x)=\cos(k\arccos x)$ and, for $x\in\mathbb{C}$, it follows by analytic continuation. It is interesting to note that, up to a linear change of variable, $\{x^n\}$ and the Chebyshev polynomials are the unique families of polynomials that satisfy an identity similar to (2) (see [2, Chapter 4, Theorem 4.4] for details); in particular, similar inversion formulas can be given for expansions in $\{x^n\}$ on $[0,1]$.

Prior to continuing, let us establish our notation. For $x\in[-1,1]$, we have $T_n(x)=\cos(n\arccos x)$ so $T_n:[-1,1]\to[-1,1]$. But we can also consider T_n both as $T_n:\mathbb{R}\to\mathbb{R}$ or $T_n:\mathbb{C}\to\mathbb{C}$. Moreover, $T_n(x)\in\mathbb{Z}[x]$, so we also have $T_n:\mathbb{Z}\to\mathbb{Z}$ and $T_n:\mathbb{Q}\to\mathbb{Q}$. Let us then use Δ to denote $[-1,1]$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, or $\mathbb{Q}$, accordingly. Thus, for functions f of type $f:\Delta\to\mathbb{R}$ (or $\mathbb{C}$), the composition $f(T_n(x))$ is well defined for every n. (Some perhaps more "esoteric" choices can be taken into account for Δ, such as $[1,\infty)$, $\mathbb{N}$, or the algebraic numbers.)

The main result of this paper is the following:

Theorem 1. *Let $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$ and $\Delta=[-1,1]$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, or $\mathbb{Q}$. If f is a bounded function defined on Δ, then the series*

$$g(x)=\sum_{n=1}^{\infty}\frac{1}{n^s}f\big(T_n(x)\big),\quad x\in\Delta, \tag{3}$$

is absolutely convergent, the function g is bounded, and we can recover f as

$$f(x)=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}g\big(T_n(x)\big),\quad x\in\Delta. \tag{4}$$

Conversely, if we have a bounded function g on Δ, the function f defined as in (4) *is bounded and fulfills* (3).

We also study some other conditions that yield similar results. In particular, in Section 3 we give a more general approach to our inversion formula.

2. The transform and the inversion formula: Proof of the main theorem

Let us begin by defining an operation $\odot$ similar to the $\circ$ in (1), but properly adapted to our circumstances. Given a function f on Δ and an arithmetical function $\alpha : \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$), we define the transform

$$g(x) = (\alpha \odot f)(x) = \sum_{n=1}^{\infty} \alpha(n) f\big(T_n(x)\big), \tag{5}$$

provided that the series converges.

Let us suppose that we have another arithmetical function $\beta : \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$) that is inverse to α with respect to Dirichlet convolution, i.e., $\alpha * \beta = \delta$ with $\delta(1) = 1$ and $\delta(n) = 0$ for $n > 1$. Let us calculate $(\beta \odot g)(x)$, at least formally, from (5). If the formal manipulations that follow are analytically justified, we can reorder series, group the terms such than $nm = k$, use (2), $\alpha * \beta = \delta$, and $T_1(x) = x$, so

$$\begin{aligned}
(\beta \odot g)(x) &= \sum_{n\in\mathbb{N}} \beta(n)(\alpha \odot f)\big(T_n(x)\big) \\
&= \sum_{n\in\mathbb{N}} \beta(n) \sum_{m\in\mathbb{N}} \alpha(m) f\big(T_m\big(T_n(x)\big)\big) \\
&= \sum_{n,m\in\mathbb{N}} \beta(n)\alpha(m) f\big(T_{mn}(x)\big) \\
&= \sum_{k\in\mathbb{N}} \bigg(\sum_{nm=k} \beta(n)\alpha(m)\bigg) f\big(T_k(x)\big) \\
&= \sum_{k\in\mathbb{N}} (\alpha * \beta)(k) f\big(T_k(x)\big) \\
&= f(x).
\end{aligned} \tag{6}$$

Thus, we have found the inversion formula. It remains to determine conditions under which the series that define $(\alpha \odot f)(x)$ and $(\beta \odot g)(x)$ converge and the manipulations in (6) can be justified.

Some simple assumptions guaranteeing this are the following:

Proposition 1. *Let α and β be two arithmetical functions related by $\alpha * \beta = \delta$, and such that $\sum_{n=1}^{\infty} |\alpha(n)| < \infty$ and $\sum_{n=1}^{\infty} |\beta(n)| < \infty$; let Δ be $[-1, 1]$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, or $\mathbb{Q}$. If f is a bounded function defined on Δ, then the series*

$$g(x) = \sum_{n=1}^{\infty} \alpha(n) f\big(T_n(x)\big), \quad x \in \Delta, \tag{7}$$

is absolutely convergent, the function g is bounded by

$$\sup_{x\in\Delta} \big|g(x)\big| \leqslant \bigg(\sum_{n=1}^{\infty} \big|\alpha(n)\big|\bigg) \sup_{x\in\Delta} \big|f(x)\big|, \tag{8}$$

and we can recover f as

$$f(x)=\sum_{n=1}^{\infty}\beta(n)g\big(T_n(x)\big),\quad x\in\Delta. \tag{9}$$

Conversely, if we have a bounded function g on Δ, the function f defined as in (9) *is bounded in a similar way and fulfills* (7).

With this, we have

Proof of Theorem 1. In the proposition, take $\alpha(n)=\alpha_s(n)=n^{-s}$, which is a completely multiplicative function whose inverse is $\alpha^{-1}(n)=\mu(n)n^{-s}$. As $\mathrm{Re}(s)>1$, it follows that $\sum_{n=1}^{\infty}|a(n)|=\sum_{n=1}^{\infty}|n^{-s}|=\zeta(\mathrm{Re}(s))$, where $\zeta(s)$ denote the Riemann's zeta function. The inversion part is similar. □

Another example. Let us consider the Liouville function $\lambda(n)$, defined by

$$\lambda(n)=\begin{cases}1, & \text{if } n=1,\\ (-1)^{a_1+\cdots+a_k}, & \text{if } n=p_1^{a_1}\cdots p_k^{a_k}\end{cases}$$

(where $p_1^{a_1}\cdots p_k^{a_k}$ denotes the decomposition of n into prime factors). $\lambda(n)$ is completely multiplicative whose inverse function is $\lambda^{-1}(n)=\mu(n)\lambda(n)=|\mu(n)|$. Then, in a similar way to Theorem 1, for $\mathrm{Re}(s)>1$ and bounded functions, we have

$$g(x)=\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s}f\big(T_n(x)\big),\quad x\in\Delta,$$

if and only if

$$f(x)=\sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^s}g\big(T_n(x)\big),\quad x\in\Delta.$$

3. A more general approach

The assumptions in Proposition 1 are very demanding. Here we study other general conditions under which the transformation formula holds.

For an arithmetical function ρ, we say that $f\in L(\Delta,\rho)$ if

$$\sum_{n=1}^{\infty}\big|\rho(n)f\big(T_n(x)\big)\big|<\infty,\quad \forall x\in\Delta$$

(recall that we are using Δ to denote $[-1,1]$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$ or $\mathbb{Q}$). In particular, $f\in L(\Delta,\alpha)$ means that (5) converges absolutely for every $x\in\Delta$.

Once again we use the arithmetical function δ defined by $\delta(1)=1$ and $\delta(n)=0$ for all $n>1$. The relation $\delta\odot f=f$ follows easily from $T_1(x)=x$.

Analogously to the mixed associative property between $\circ$ and Dirichlet convolution $*$, we have the following version between $\odot$ and $*$. The proof is straightforward, because the absolute convergence allows the rearrangement of the sums.

Proposition 2. *Let α, β be two arithmetical functions, $f:\Delta\to\mathbb{R}$ (or $\mathbb{C}$), and suppose at a given $x\in\Delta$,*

$$\sum_{n,m\in\mathbb{N}}\left|\alpha(n)\beta(m)f\big(T_{nm}(x)\big)\right|=\sum_{k\in\mathbb{N}}\big(|\alpha|*|\beta|\big)(k)\left|f\big(T_k(x)\big)\right|<\infty. \tag{10}$$

Then, all the series involved in the definitions of $(\alpha\odot(\beta\odot f))(x)$ and $((\alpha*\beta)\odot f)(x)$ are absolutely convergent and

$$\big(\alpha\odot(\beta\odot f)\big)(x)=\big((\alpha*\beta)\odot f\big)(x).$$

In particular, if $f\in L(\Delta,|\alpha|*|\beta|)$, then $\alpha\odot(\beta\odot f)=(\alpha*\beta)\odot f$.

In this general context, the inversion formula becomes

Proposition 3. *Let α be an arithmetical function with Dirichlet convolution inverse α^{-1}. Given a function $f:\Delta\to\mathbb{R}$ (or $\mathbb{C}$), with $f\in L(\Delta,|\alpha|*|\alpha^{-1}|)$, the transform $g(x)=(\alpha\odot f)(x)$ is defined for all $x\in\Delta$. Moreover, if $g\in L(\Delta,|\alpha|*|\alpha^{-1}|)$, then $f(x)=(\alpha^{-1}\odot g)(x)$ for all $x\in\Delta$.*

Proof. By Proposition 2,

$$\alpha^{-1}\odot g=\alpha^{-1}\odot(\alpha\odot f)=\big(\alpha^{-1}*\alpha\big)\odot f=\delta\odot f=f.$$

For the second part, recall that $|\alpha|*|\alpha^{-1}|=|\alpha^{-1}|*|\alpha|$. □

In general, it does not seem easy to check that the condition $f\in L(\Delta,|\alpha|*|\alpha^{-1}|)$ implies $g\in L(\Delta,|\alpha|*|\alpha^{-1}|)$; this—if true—would mean that the inversion formula $\alpha^{-1}\odot g$ is defined without this extra hypothesis.

The following special case of Proposition 3 has special interest:

Proposition 4. *Let α be a completely multiplicative arithmetical function, $f:\Delta\to\mathbb{R}$ (or $\mathbb{C}$), and suppose that $f\in L(\Delta,\alpha d)$ (where $d(n)$ is the number of divisors of n). Then*

$$g(x)=\sum_{n\in\mathbb{N}}\alpha(n)f\big(T_n(x)\big)$$

is defined for all $x\in\Delta$. Moreover, if $g\in L(\Delta,\alpha d)$, then

$$f(x)=\sum_{n\in\mathbb{N}}\mu(n)\alpha(n)g\big(T_n(x)\big)$$

for all $x\in\Delta$.

Proof. If α is completely multiplicative, then $\alpha^{-1}(n)=\mu(n)\alpha(n)$. Moreover,

$$\big(|\alpha|*\left|\alpha^{-1}\right|\big)(k)=\sum_{nm=k}\left|\alpha(n)\mu(m)\alpha(m)\right|\leqslant\sum_{nm=k}\left|\alpha(nm)\right|=d(k)|\alpha|(k)=\left|d(k)\alpha(k)\right|,$$

so the hypothesis $f\in L(\Delta,\alpha d)$ allows us to apply Proposition 3. The same holds with respect to $g\in L(\Delta,\alpha d)$. □

Remark. As commented previously, it does not seem easy to check if $g\in L(\Delta,\alpha d)$ given that $f\in L(\Delta,\alpha d)$. However, we claim that something weaker is true:

$$f\in L\big(\Delta,\alpha d^2\big)\quad\Rightarrow\quad g\in L(\Delta,\alpha d).$$

To prove this, take into account that $d(n) \leqslant d(k)$ when $n \mid k$, and notice also that $|\alpha|$ is completely multiplicative. Thus

$$\begin{aligned}\sum_{n\in\mathbb{N}} d(n)\left|\alpha(n)g\left(T_n(x)\right)\right| &= \sum_{n\in\mathbb{N}} d(n)\left|\alpha(n)\right|\left|\sum_{m\in\mathbb{N}} \alpha(m) f\left(T_m\left(T_n(x)\right)\right)\right| \\ &\leqslant \sum_{k\in\mathbb{N}} d(k)\left(\sum_{nm=k}\left|\alpha(n)\alpha(m)\right|\right)\left|f\left(T_k(x)\right)\right| \\ &= \sum_{k\in\mathbb{N}} d(k)^2\left|\alpha(k)\right|\left|f\left(T_k(x)\right)\right| < \infty\end{aligned}$$

since $f \in L(\Delta, \alpha d^2)$, so the claim is proved. Actually, the extra factor $d(n)$ is not very troublesome, because $d(n) = o(n^r)$ for every $r > 0$ (see [1, Section 18.1, Theorem 315, p. 260]).

References

[1] G.M. Hardy, E.M. Wright, An Introduction to the Theory of Numbers, fifth ed., Oxford Univ. Press, 1979.

[2] T.J. Rivlin, Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory, second ed., Wiley, 1990.