

# The Möbius inversion formula for Fourier series applied to Bernoulli and Euler polynomials

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## Abstract

Hurwitz found the Fourier expansion of the Bernoulli polynomials over a century ago. In general, Fourier analysis can be fruitfully employed to obtain properties of the Bernoulli polynomials and related functions in a simple manner. In addition, applying the technique of Möbius inversion from analytic number theory to Fourier expansions, we derive identities involving Bernoulli polynomials, Bernoulli numbers, and the Möbius function; this includes formulas for the Bernoulli polynomials at rational arguments. Finally, we show some asymptotic properties concerning Bernoulli and Euler polynomials.

*Key words:* Bernoulli polynomials, Euler polynomials, Fourier series, Möbius transform, inversion formula, rational arguments, asymptotic properties

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## 1. Introduction

The Bernoulli polynomials, which play an important role in Analytic Number Theory, are usually defined by means of the generating function

$$\frac{te^{tx}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}.$$

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The polynomial  $B_k(x)$  is monic and has degree  $k$ . For example,  $B_0(x) = 1$ ,  $B_1(x) = x - \frac{1}{2}$ . Although there are no closed formulas for the  $k$ th Bernoulli polynomial, the uniqueness theorem for power series expansions allows one to easily prove various properties regarding them. Among these we recall the following:

$$B'_{k+1}(x) = (k+1)B_k(x) \quad (1.1)$$

and

$$B_k(1-x) = (-1)^k B_k(x). \quad (1.2)$$

In 1890, Hurwitz found the Fourier expansions

$$B_{2k}(x) = \frac{2(-1)^{k-1}(2k)!}{(2\pi)^{2k}} \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{n^{2k}}, \quad x \in [0, 1), \quad k \geq 1; \quad (1.3)$$

$$B_{2k+1}(x) = \frac{2(-1)^{k-1}(2k+1)!}{(2\pi)^{2k+1}} \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n^{2k+1}}, \quad x \in [0, 1), \quad k \geq 0. \quad (1.4)$$

The Bernoulli numbers are given by  $B_k = B_k(0)$ . For odd indexes, we have

$$B_{2k+1}(0) = 0, \quad k \geq 1. \quad (1.5)$$

An immediate consequence of the Fourier expansions for the Bernoulli polynomials, probably the most well-known, is the connection between the Bernoulli numbers and the values of the Riemann zeta function at the positive even integers. Indeed, one only needs to set  $x = 0$  in the first formula to obtain

$$B_{2k} = \frac{2(-1)^{k-1}(2k)!}{(2\pi)^{2k}} \zeta(2k).$$

The following formula, using Möbius inversion, is also well-known:

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2k}} = \frac{1}{\zeta(2k)} \iff 1 = \frac{(2\pi)^{2k}}{2(-1)^{k-1}(2k)!} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2k}} B_{2k}.$$

Indeed, Möbius inversion shows  $1 = \zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$  for any complex  $s$  with  $\operatorname{Re}(s) > 1$ . However, leaving  $s = 2k$  fixed, we can instead generalize the formula to

$$\cos(2\pi x) = \frac{(2\pi)^{2k}}{2(-1)^{k-1}(2k)!} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2k}} B_{2k}(\{nx\}), \quad (1.6)$$

where  $\{x\}$  denotes the fractional part of  $x$ . This expression, which provides a nice approximation to the cosine by means of polynomials, is a consequence of a more general type of Möbius inversion, discussed by two of the authors in [10]. In this case, it involves the inversion of Fourier series. The analogous result for odd Bernoulli polynomials also holds. In addition, we find a similar formula involving Euler polynomials. All of this is discussed in Section 2.

In Section 3, we consider the relations obtained by evaluating (1.3) and its Möbius inverse (1.6) at rational values. One obtains, in an elementary manner, expressions for combinations of sums of the form

$$\sum_{n \equiv r \pmod{m}} \frac{1}{n^{2k}}, \quad \sum_{n \equiv r \pmod{m}} \frac{\mu(n)}{n^{2k}}$$

(where  $n \equiv r \pmod{m}$  is a shorthand for  $n \equiv r \pmod{m}$ ) in terms of values of the Bernoulli polynomials at these rational arguments.

In Section 4, we obtain the asymptotic behavior of the Bernoulli polynomials on  $[0, 1]$  from formulas (1.3) and (1.4). This result is not new, however, our proof is simpler, and in addition we also study the rate of convergence.

Throughout the paper, we will use  $\lfloor x \rfloor$  to denote the integer part (also known as floor) of  $x \in \mathbb{R}$  (i.e.,  $\lfloor x \rfloor$  is the largest integer  $\leq x$ ); then, the fractional part of  $x$  will be  $\{x\} = x - \lfloor x \rfloor$ .

## 2. Möbius inversion formulas

### 2.1. Arithmetical Möbius inversion

In Number Theory, an *arithmetical function* is typically simply a function  $\alpha : \mathbb{N} \rightarrow \mathbb{R}$  (or  $\mathbb{C}$ ). An arithmetical function  $\alpha$  is *completely multiplicative* if it satisfies  $\alpha(nm) = \alpha(n)\alpha(m)$  for all  $n, m \in \mathbb{N}$  and is not the zero function. Given two arithmetical functions  $\alpha$  and  $\beta$ , their *Dirichlet convolution* (also called *Dirichlet product*)  $\alpha * \beta$  is defined by

$$\alpha * \beta(n) = \sum_{d|n} \alpha(d)\beta\left(\frac{n}{d}\right) = \sum_{d|n} \alpha\left(\frac{n}{d}\right)\beta(d) = \sum_{ab=n} \alpha(a)\beta(b),$$

where  $\sum_{d|n}$  represents the sum over all divisors  $d$  of  $n$ . Dirichlet convolution is a commutative and associative operation on arithmetical functions, with identity the delta function at 1, i.e.  $\delta(1) = 1$  and  $\delta(n) = 0$  if  $n \neq 1$ . An arithmetical function  $\alpha$  is invertible with respect to Dirichlet convolution if and only if  $\alpha(1) \neq 0$ . In this case the unique function  $\beta$  such that  $\alpha * \beta = \beta * \alpha = \delta$  is referred to as the (Dirichlet) inverse of  $\alpha$ , and we use the standard notation  $\alpha^{-1}$  to denote it.

A fundamental role in the theory is played by the *Möbius function*  $\mu$ , which is the Dirichlet inverse of the constant function 1. It is given by

$$\begin{cases} \mu(1) = 1, \\ \mu(n) = 0 \text{ if } n \text{ has a squared factor,} \\ \mu(p_1 p_2 \cdots p_k) = (-1)^k \text{ when } p_1, p_2, \dots, p_k \text{ are distinct primes.} \end{cases}$$

For a general invertible arithmetical function  $\alpha$ , its Dirichlet inverse may be computed recursively, but it is often difficult to deduce a simple closed expression for  $\alpha^{-1}$ . An easy exception is when  $\alpha$  is completely multiplicative, since then  $\alpha^{-1} = \mu\alpha$  (the pointwise product).

The *Möbius Inversion Formula* most often refers to the equivalence  $\beta = \alpha * 1 \Leftrightarrow \alpha = \beta * \mu$ , which is an immediate algebraic consequence of the facts stated above, and which

may be written explicitly as

$$\beta(n) = \sum_{d|n} \alpha(d) \iff \alpha(n) = \sum_{d|n} \mu(n/d)\beta(d).$$

## 2.2. Möbius inversion of Fourier series

For our purposes, we need a variation on the inversion theme of a more analytic nature, belonging to a class of formulas also referred to as Möbius inversion. We shall restrict ourselves to the case of Fourier series. Suppose we have a real-variable function  $f$  expanded in a Fourier series,

$$f(x) = \sum_{n=1}^{\infty} \alpha(n)e^{2\pi inx}.$$

Regard the Fourier coefficients  $\alpha(n)$  as an arithmetical function. Now consider any arithmetical function  $\beta$ , and form the “generalized convolution”

$$(\beta \odot f)(x) = \sum_{m=1}^{\infty} \beta(m)f(mx).$$

Substituting the Fourier series for  $f$  into this expression, one finds formally that

$$\begin{aligned} (\beta \odot f)(x) &= \sum_{m=1}^{\infty} \beta(m) \sum_{n=1}^{\infty} \alpha(n)e^{2\pi inmx} = \sum_{m,n=1}^{\infty} \alpha(n)\beta(m)e^{2\pi inmx} \\ &= \sum_{l=1}^{\infty} \left( \sum_{mn=l} \alpha(n)\beta(m) \right) e^{2\pi ilx} = \sum_{l=1}^{\infty} (\alpha * \beta)(l)e^{2\pi ilx}. \end{aligned}$$

Thus  $\beta \odot f$  is the Fourier series with coefficients given by the Dirichlet convolution of  $\alpha$  and  $\beta$ . Now, if  $\alpha$  is invertible and we take  $\beta = \alpha^{-1}$ , so that  $\alpha * \beta = \delta$ , this reduces to

$$e^{2\pi ix} = (\alpha^{-1} \odot f)(x), \tag{2.1}$$

so we have an expansion of the exponential in terms of the function  $f$  whose Fourier series we started out with. Note also that the Fourier series itself is the generalized convolution of the Fourier coefficients with the function  $E(x) = e^{2\pi ix}$ , namely  $f = \alpha \odot E$ . In fact formally one has in general that  $\alpha \odot (\beta \odot g) = (\alpha * \beta) \odot g$  for any function  $g$  and arithmetical functions  $\alpha, \beta$ , and the inversion relation  $f = \alpha \odot g \iff g = \alpha^{-1} \odot f$ .

*Remark 1.* Justifying the formal steps above is not hard in the case of a bounded function such as  $e^{2\pi ix}$ . It is enough for the double series to converge absolutely, which is implied by  $\sum_{m,n=1}^{\infty} |\alpha(n)\beta(m)| < \infty$ . Note that this is equivalent to  $\sum_{l=1}^{\infty} (|\alpha| * |\beta|)(l) < \infty$ . In particular, the inversion formula (2.1) holds if  $\sum_{l=1}^{\infty} (|\alpha| * |\alpha^{-1}|)(l) < \infty$ . In the case of a completely multiplicative function  $\alpha$ , since  $\alpha^{-1} = \mu\alpha$  and  $|\mu| \leq 1$ , it is sufficient to check if  $\sum_{l=1}^{\infty} (|\alpha| * |\alpha|)(l) < \infty$ .

By taking real and imaginary parts, we obtain analogous formulas involving the functions  $\sin(2\pi x)$  and  $\cos(2\pi x)$ . In this guise, the idea of applying Möbius inversion to

Fourier series goes at least as far back as Chebyshev [8] and appears recently in [9] in a study of a lattice problem in Physics.

Let us mention that the above results are special cases of a general theory which extends far beyond the case of Fourier series and which seems to originate with a little-known idea of Cesàro [7], rediscovered on occasion, for example in [6]. The interested reader may consult [4] for an abstract formulation, [5] for a series of concrete examples, [10] for an inversion formula involving Chebyshev polynomials, and [2] as a general reference for analytic number theory.

If we want to obtain concrete approximation results from formulas such as (2.1), we need to have an expression for the Dirichlet inverse of the Fourier coefficients that we can work with, and the best case of this occurs when they are completely multiplicative. Now, this certainly does not happen in general. For this reason, it is an interesting question to determine which functions do indeed give completely multiplicative Fourier coefficients, at least modulo constant factors. This happens, for instance, in the case of the square and triangular waves, which were the examples studied by Chebyshev. Perhaps surprisingly, this also happens with a well-known family of functions: the Bernoulli polynomials.

### 2.3. Möbius inversion of the Fourier series of the Bernoulli polynomials

The Bernoulli polynomials  $B_k(x)$  play an important role in various expansions and approximation formulas which are useful both in analytic number theory and in classical and numerical analysis. These polynomials can be defined by various methods depending on the applications (see [12] and the references therein).

The Fourier expansions (1.3) and (1.4) are actually valid for  $0 \leq x \leq 1$ , and the convergence is absolute and uniform on  $[0, 1]$ , except for  $B_1(x)$  that requires  $0 < x < 1$ . For the time being, we disregard  $B_1(x)$  and also  $B_0(x)$ . From (1.5) and (1.2), it is clear that

$$B_k(0) = B_k(1), \quad k \geq 2,$$

so we can construct the periodic extension of  $B_k(x)$  on  $[0, 1]$  to  $\mathbb{R}$  by taking fractional parts  $\{x\} = x - \lfloor x \rfloor$  and using  $B_k(\{x\})$  instead of  $B_k(x)$ ; these  $[0, 1]$ -periodic extensions are continuous. Then, (1.3) and (1.4) for  $k \geq 1$  become

$$B_{2k}(\{x\}) = \frac{2(-1)^{k-1}(2k)!}{(2\pi)^{2k}} \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{n^{2k}}, \quad x \in \mathbb{R}, \quad (2.2)$$

$$B_{2k+1}(\{x\}) = \frac{2(-1)^{k-1}(2k+1)!}{(2\pi)^{2k+1}} \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n^{2k+1}}, \quad x \in \mathbb{R}. \quad (2.3)$$

Now, applying the real versions of (2.1), we obtain the following.

**Theorem 2.1.** *For every  $k \geq 1$ , the functions cosine and sine expand in terms of the Bernoulli polynomials  $B_{2k}$  and  $B_{2k+1}$ , respectively, as*

$$\cos(2\pi x) = \frac{(-1)^{k-1}(2\pi)^{2k}}{2(2k)!} \sum_{n=1}^{\infty} \frac{\mu(n)B_{2k}(\{nx\})}{n^{2k}}, \quad x \in \mathbb{R}, \quad (2.4)$$

$$\sin(2\pi x) = \frac{(-1)^{k-1}(2\pi)^{2k+1}}{2(2k+1)!} \sum_{n=1}^{\infty} \frac{\mu(n)B_{2k+1}(\{nx\})}{n^{2k+1}}, \quad x \in \mathbb{R}. \quad (2.5)$$

*Proof.* This is a special case of (2.1). Up to constants, we are dealing with the arithmetical functions  $\alpha_s(n) = n^{-s}$ , which are completely multiplicative for any  $s \in \mathbb{C}$  (here  $s = 2k$  or  $2k + 1$ ). Hence  $\alpha_s^{-1} = \mu\alpha_s$ . The constants simply affect inversion by  $(c\alpha_s)^{-1} = c^{-1}\mu\alpha_s$  and of course do not affect the convergence that justifies the inversion. As for the convergence itself, by Remark 1, it is enough to show that  $\sum_{n=1}^{\infty} (|\alpha_s| * |\alpha_s|)(n) < \infty$ . This is true whenever  $\sigma = \operatorname{Re}(s) > 1$ , since  $|\alpha_s(n)| = \alpha_\sigma(n)$  and  $(\alpha_\sigma * \alpha_\sigma)(n) = \sum_{k|n} k^{-\sigma} (n/k)^{-\sigma} = \sum_{k|n} n^{-\sigma} = d(n)n^{-\sigma}$ , where  $d(n)$  is the number of divisors of  $n$ . A standard result from Analytic Number Theory states that  $d(n) = o(n^\epsilon)$  for any  $\epsilon > 0$ . Hence  $(\alpha_\sigma * \alpha_\sigma)(n) = O(n^{-\sigma+\epsilon})$  and so  $\sum_{n=1}^{\infty} (\alpha_\sigma * \alpha_\sigma)(n)$  converges by comparison with the zeta series.  $\square$

#### 2.4. Special values

In Analytic Number Theory one often obtains interesting arithmetical results by evaluating relations involving transcendental functions at rational arguments. Since  $e^{2\pi ix}$  is a root of unity, hence an algebraic number, when  $x$  is rational, its imaginary part  $\sin(2\pi x)$  is also algebraic. A “nice” expression for this algebraic number exists, for example, when it is constructible, in the sense of Galois Theory, and the most famous case of this is  $x = 1/17$ , corresponding to the construction of the regular 17-gon found by Gauss. Algebraically this means that  $\cos(2\pi/17)$  has an expression in nested square roots. Applying the first formula of Theorem 2.1 to  $x = 1/17$  and, for simplicity, to the lowest valid value of  $k$ , that is  $k = 1$ , yields, after evaluating  $B_2(r/17)$ ,

$$\begin{aligned} -1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + 2\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}} \\ = \frac{2^3\pi^2}{3 \cdot 17^2} \sum_{n=1}^{\infty} \frac{\mu(n)\beta(n)}{n^2} \end{aligned}$$

where

$$\beta(n) = \begin{cases} 289, & \text{if } n \equiv 0 \pmod{17}, \\ 193, & \text{if } n \equiv \pm 1 \pmod{17}, \\ 109, & \text{if } n \equiv \pm 2 \pmod{17}, \\ 37, & \text{if } n \equiv \pm 4 \pmod{17}, \\ -23, & \text{if } n \equiv \pm 4 \pmod{17}, \\ -71, & \text{if } n \equiv \pm 5 \pmod{17}, \\ -107, & \text{if } n \equiv \pm 6 \pmod{17}, \\ -131, & \text{if } n \equiv \pm 7 \pmod{17}, \\ -143, & \text{if } n \equiv \pm 8 \pmod{17}. \end{cases}$$

This is the kind of “explicit formula” one can obtain with these methods. Note the amount and variety of mathematics that goes into this result: the ideas of Bernoulli, Cesàro, Chebyshev, Dirichlet, Euler, Fourier, Galois, Gauss, Hurwitz and Möbius are all involved.

### 2.5. The case of Euler polynomials

In a way similar to their cousins the Bernoulli polynomials, the Euler polynomials are defined by means of the generating function

$$\frac{2e^{2tx}}{e^t + 1} = \sum_{k=0}^{\infty} E_k(x) \frac{t^k}{k!},$$

which is convergent for  $|t| < \pi$ .

For  $0 \leq x \leq 1$  ( $0 < x < 1$  in the case of  $E_0(x)$ ), the Euler polynomials have Fourier expansions also similar to those of the Bernoulli polynomials:

$$E_{2k-1}(x) = \frac{4(-1)^k(2k-1)!}{\pi^{2k}} \sum_{n=0}^{\infty} \frac{\cos((2n+1)\pi x)}{(2n+1)^{2k}}, \quad k \geq 1, \quad (2.6)$$

$$E_{2k}(x) = \frac{4(-1)^k(2k)!}{\pi^{2k+1}} \sum_{n=0}^{\infty} \frac{\sin((2n+1)\pi x)}{(2n+1)^{2k+1}}, \quad k \geq 0. \quad (2.7)$$

Let us find the Möbius inverse of these series. By denoting

$$\alpha_k(j) = \begin{cases} 0 & \text{if } j \text{ is even,} \\ \frac{1}{(2n+1)^k} & \text{if } j \text{ is odd, } j = 2n+1, \end{cases}$$

we can write  $E_{2k-1}(x)$  in (2.6) as a constant times  $\sum_{j=1}^{\infty} \alpha_{2k}(j) \cos(j\pi x)$ , and similarly for (2.7). Moreover, the function  $\alpha_k$  is completely multiplicative, so its Dirichlet inverse is  $\alpha_k^{-1} = \mu \alpha_k$ .

Extending  $E_k(x)$  from  $[0, 1]$  to  $\mathbb{R}$  is only a bit more complicated than for  $B_k(x)$ . The extension that is compatible with (2.6) and (2.7) is  $(-1)^{\lfloor x \rfloor} E_k(\{x\})$ . Then, in a manner entirely similar to Theorem 2.1, we deduce the following.

**Theorem 2.2.** *For every  $k \geq 1$ , the functions cosine and sine expand in terms of the Euler polynomials  $E_{2k-1}$  and  $E_{2k}$ , respectively, as*

$$\begin{aligned} \cos(\pi x) &= \frac{(-1)^k \pi^{2k}}{4(2k-1)!} \sum_{n=0}^{\infty} \frac{\mu(2n+1)(-1)^{\lfloor (2n+1)x \rfloor} E_{2k-1}(\{(2n+1)x\})}{(2n+1)^{2k}}, \quad x \in \mathbb{R}, \\ \sin(\pi x) &= \frac{(-1)^k \pi^{2k+1}}{4(2k)!} \sum_{n=0}^{\infty} \frac{\mu(2n+1)(-1)^{\lfloor (2n+1)x \rfloor} E_{2k}(\{(2n+1)x\})}{(2n+1)^{2k+1}}, \quad x \in \mathbb{R}. \end{aligned}$$

### 3. Sums of restricted zeta series and their Möbius inverses

The evaluation of (2.2), (2.3) and their inverses (2.4), (2.5) at rational arguments  $x = r/m$  introduces a periodicity modulo  $m$  into the sums which on rearrangement by residue classes causes the following sums to appear:

$$M_m(k, r) = \sum_{n \equiv r \pmod{m}} \frac{\mu(n)}{n^k}, \quad Z_m(k, r) = \sum_{n \equiv r \pmod{m}} \frac{1}{n^k}, \quad r = 0, 1, \dots, m-1, \quad (3.1)$$

where  $k, m \in \mathbb{N}$ ,  $k, m \geq 2$  and the sums are always over positive integers (thus the sum defining  $M_m(k, 0)$  begins at  $n = m$ ).

These sums are related to the Prime Number Theorem for arithmetic progressions. Using techniques from Analytic Number Theory, the sums  $Z$  and  $M$  can be expressed in terms of  $L$ -series for Dirichlet characters. Here we show that, starting from the Fourier expansions of the Bernoulli polynomials and their Möbius inverses, certain sums and differences of  $M$  and  $Z$  over a symmetric pair  $\pm r$  of residue classes modulo  $m$  can be evaluated explicitly by elementary and computationally feasible means, using only linear algebra, as a consequence of another auxiliary result involving matrices defined by the Bernoulli polynomials  $B_k$  and the cosine function. This approach is in the spirit of one of the problems dealt with in [11], which studies the case  $k = 1$ , and, as is amply discussed there, is interesting in its own right.

We concentrate on the case of an even power  $2k$  mostly, since we make use of the evaluation of  $\zeta(2k)$  in several places, but in the last part of this section we also derive some results for an odd power  $2k + 1$ .

The methods employed in this section may also be used to derive results for Euler polynomials and Euler numbers which are analogous to those we obtain for Bernoulli polynomials and Bernoulli numbers. We have chosen to illustrate the method with the latter to allow an easier comparison with results in the literature, and for reasons of space the corresponding formulas for Euler polynomials are left to the reader.

### 3.1. Linear relations among values at rational arguments

Trivially, we have

$$\sum_{r=0}^{m-1} M_m(k, r) = \frac{1}{\zeta(k)}, \quad \sum_{r=0}^{m-1} Z_m(k, r) = \zeta(k), \quad Z_m(k, 0) = \frac{1}{m^k} \zeta(k).$$

Let us introduce notation for the constants which appear in the Fourier expansions of the Bernoulli polynomials,

$$C(k) = \frac{(-1)^{k-1} (2\pi)^{2k}}{2(2k)!}, \quad D(k) = \frac{(-1)^{k-1} (2\pi)^{2k+1}}{2(2k+1)!}.$$

Fixing  $k$  and  $m$ , let

$$\omega = \lfloor m/2 \rfloor$$

and define, for odd  $m$ ,

$$\begin{aligned} x_i &= C(k)(M_m(2k, i) + M_m(2k, m - i)), \quad i = 1, \dots, \omega, \\ x_{\omega+1} &= C(k)M_m(2k, 0), \\ y_i &= C(k)^{-1}(Z_m(2k, i) + Z_m(2k, m - i)), \quad i = 1, \dots, \omega, \\ y_{\omega+1} &= C(k)^{-1}Z_m(2k, 0), \end{aligned}$$

and for even  $m$ , the same expressions except that at  $i = \omega = m/2$  we take

$$x_{m/2} = C(k)M_m(2k, m/2), \quad y_{m/2} = C(k)^{-1}Z_m(2k, m/2)$$

and not twice this expression as the previous formulas would indicate. Note that in fact  $\zeta(2k) = C(k)B_{2k}$ , and hence

$$y_{\omega+1} = B_{2k}/m^{2k}. \tag{3.2}$$



**Proposition 3.1.** *Let  $m \in \mathbb{N}$ ,  $m \geq 2$ , and  $k \in \mathbb{N}$ . Then*

$$\cos(2\pi r/m) = \sum_{j=1}^{\omega} B_{2k} \left( \left\{ \frac{jr}{m} \right\} \right) x_j + B_{2k} x_{\omega+1}, \quad r = 0, 1, \dots, \omega, \quad (3.3)$$

$$B_{2k}(r/m) = \sum_{j=1}^{\omega} \cos \left( 2\pi \frac{jr}{m} \right) y_j + y_{\omega+1}, \quad r = 0, 1, \dots, \omega. \quad (3.4)$$

*Proof.* (3.3) follows immediately by evaluating (2.4) at the  $m$  arguments  $x = r/m$ ,  $r = 0, \dots, m-1$ , grouping the series by residues modulo  $m$  and taking into account the symmetry  $B_{2k}(1-x) = B_{2k}(x)$ . (3.4) is obtained in exactly the same way via (2.2) and  $\cos(2\pi(1-x)) = \cos(2\pi x)$ .  $\square$

*Remark 2.* The equations (3.4) give formulas for the values of Bernoulli polynomials at rational arguments, but they are not very satisfactory as they involve the terms  $y_i$  for which there are no simple expressions (with the exception of  $y_{\omega+1} = B_{2k}/m^{2k}$ ). In fact, (3.4) are practically the content of the paper [13], where the matter is not taken any further; a posterior paper that deals with this matter and related topics is [15]. Here we are going to show some of their applications, as well as those of their ‘‘Möbius inverses’’ (3.3).

### 3.2. Evaluation of $M_m(2k, 0)$ by elementary methods

The next step is to show that one can evaluate  $x_{\omega+1} = C(k)M_m(2k, 0)$  explicitly. This can be done with Dirichlet series and characters, but in fact it is not too difficult to give a nice alternative elementary proof, as we proceed to show.

**Theorem 3.2.** *Let  $m = p_1 \cdot p_2 \cdots p_l$  where the  $p_i$  are distinct primes, and let  $k \in \mathbb{N}$ . Then*

$$x_{\omega+1} = \frac{(-1)^{k-1} (2\pi)^{2k}}{2(2k)!} \sum_{n=1}^{\infty} \frac{\mu(mn)}{(mn)^{2k}} = \frac{1}{B_{2k}} \prod_{i=1}^l \frac{1}{1 - p_i^{2k}}. \quad (3.5)$$

If  $m$  is not squarefree,  $x_{\omega+1} = 0$  trivially since  $\mu(mn) = 0$  for all  $n \in \mathbb{N}$ .

*Proof.* The statement is equivalent to  $M_m(2k, 0) = \zeta(2k)^{-1} \prod_{i=1}^l (1 - p_i^{2k})^{-1}$ . We shall prove a general formula

$$\sum_{n \equiv 0 \pmod{m}} \frac{\mu(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\mu(mn)}{(mn)^s} = \frac{\zeta(s)^{-1}}{\prod_{i=1}^l (1 - p_i^s)}$$

for any complex number  $s$  with  $\operatorname{Re}(s) > 1$ . Consider everything fixed except  $m$  and denote this sum by  $S(m)$ . Note that, by definition of  $\mu$ ,

$$\mu(mn) = \begin{cases} (-1)^l \mu(n) & \text{if no } p_i \text{ divides } n, \\ 0 & \text{if some } p_i \text{ divides } n. \end{cases}$$

For  $l = 1$ , the case where  $m$  is equal to a prime  $p$ , the formula simply states  $S(p) = \zeta(s)^{-1} (1 - p^s)^{-1}$  and we have  $\mu(pn) = -\mu(n)$  or  $0$  according to  $n \not\equiv 0 \pmod{p}$  or  $n \equiv 0 \pmod{p}$ .

Thus

$$\begin{aligned} S(p) &= -\frac{1}{p^s} \sum_{n \neq 0 \pmod{p}} \frac{\mu(n)}{n^s} = -\frac{1}{p^s} \left( \sum_n \frac{\mu(n)}{n^s} - \sum_{n \equiv 0 \pmod{p}} \frac{\mu(n)}{n^s} \right) \\ &= -\frac{1}{p^s} (\zeta(s)^{-1} - S(p)), \end{aligned}$$

and hence  $S(p) = \zeta(s)^{-1}(1 - p^s)^{-1}$  follows immediately.

The general case can be proved by induction on  $l$ , applying the inclusion-exclusion formula. In general,  $S(p_1 p_2 \cdots p_l)$  will appear as a combination of itself and all the sums  $S(p_{i_1} p_{i_2} \cdots p_{i_t})$  with distinct indices  $i_j$  and  $0 \leq t < l$ . For example, for  $l = 2$ , if  $m = pq$  with  $p, q$  distinct primes, we have

$$S(pq) = \frac{1}{p^s q^s} \sum_{n \neq 0 \pmod{p}, n \neq 0 \pmod{q}} \frac{\mu(n)}{n^s} = \frac{1}{p^s q^s} (\zeta(s)^{-1} - S(p) - S(q) + S(pq)),$$

hence, substituting the values for  $S(p)$  and  $S(q)$ , we can solve for  $S(pq)$ , yielding the corresponding formula.  $\square$

### 3.3. Solution of the “Bernoulli system” of linear equations

Once we have elementary formulas for the terms  $x_{\omega+1}$  (Theorem 3.2) and  $y_{\omega+1}$  (equation (3.2)), the system of equations (3.3) and (3.4) may be considered as involving only the unknowns  $x_j$  and  $y_j$  for  $j = 1, \dots, \omega$ ,

$$\cos(2\pi r/m) - B_{2k} x_{\omega+1} = \sum_{j=1}^{\omega} B_{2k} \left( \left\{ \frac{jr}{m} \right\} \right) x_j, \quad r = 1, \dots, \omega, \quad (3.6)$$

$$B_{2k}(r/m) - \frac{B_{2k}}{m^{2k}} = \sum_{j=1}^{\omega} \cos \left( 2\pi \frac{jr}{m} \right) y_j, \quad r = 1, \dots, \omega. \quad (3.7)$$

The associated matrices will be denoted by

$$\mathbf{B}_{2k,m} = \left( B_{2k} \left( \left\{ \frac{ij}{m} \right\} \right) \right)_{i,j=1}^{\omega}, \quad \mathbf{Cos}_m = \left( \cos \left( 2\pi \frac{ij}{m} \right) \right)_{i,j=1}^{\omega}. \quad (3.8)$$

Let us show, with a small caveat, that these matrices are regular. Indeed we compute the determinant of the cosine matrix explicitly in closed form.

**Theorem 3.3.** *Let  $m \in \mathbb{N}$ . Then*

$$\begin{aligned} \det(\mathbf{Cos}_{2m+1}) &= (-1)^{\lfloor \frac{m+1}{2} \rfloor} \frac{(2m+1)^{(m-1)/2}}{2^m}, \\ \det(\mathbf{Cos}_{2m}) &= (-1)^{\lfloor \frac{m+1}{2} \rfloor} \left( \frac{m}{2} \right)^{(m-1)/2}. \end{aligned}$$

*In the odd case, if  $\mathbf{U}$  is the square matrix of order  $m$  whose entries are all 1, then*

$$\mathbf{Cos}_{2m+1}^{-1} = \frac{4}{2m+1} (-\mathbf{U} + \mathbf{Cos}_{2m+1});$$

and, in the even case,  $\mathbf{Cos}_{2m}^{-1} = (a_{i,j})$ , where

$$a_{i,j} = \begin{cases} \frac{2}{m}(-1 + \cos(\pi ij/m)), & \text{if } i, j < m, \\ \frac{1}{m}(-1 + \cos(\pi ij/m)), & \text{if } j < i = m \text{ or } i < j = m, \\ \frac{1}{2m}(-1 + \cos(\pi ij/m)), & \text{if } i = j = m. \end{cases}$$

*Proof.* We assume  $m$  is odd; the even case is similar. Replace the last row with the sum of every row and move it up to the first row. It is easy to show that

$$\sum_{r=1}^{\omega} \cos\left(2\pi \frac{jr}{m}\right) = \begin{cases} -1/2, & \text{if } m \text{ is odd,} \\ ((-1)^j - 1)/2, & \text{if } m \text{ is even,} \end{cases}$$

hence we obtain

$$\begin{aligned} & \begin{vmatrix} \cos\left(\frac{2\pi}{2m+1}\right) & \cos\left(\frac{4\pi}{2m+1}\right) & \cdots & \cos\left(\frac{2m\pi}{2m+1}\right) \\ \cdots & \cdots & \cdots & \cdots \\ \cos\left(\frac{2(m-1)\pi}{2m+1}\right) & \cos\left(\frac{4(m-1)\pi}{2m+1}\right) & \cdots & \cos\left(\frac{2m(m-1)\pi}{2m+1}\right) \\ \cos\left(\frac{2m\pi}{2m+1}\right) & \cos\left(\frac{4m\pi}{2m+1}\right) & \cdots & \cos\left(\frac{2m^2\pi}{2m+1}\right) \end{vmatrix} \\ &= -\frac{1}{2}(-1)^{m-1} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \cos\left(\frac{2\pi}{2m+1}\right) & \cos\left(\frac{4\pi}{2m+1}\right) & \cdots & \cos\left(\frac{2m\pi}{2m+1}\right) \\ \cdots & \cdots & \cdots & \cdots \\ \cos\left(\frac{2(m-1)\pi}{2m+1}\right) & \cos\left(\frac{4(m-1)\pi}{2m+1}\right) & \cdots & \cos\left(\frac{2m(m-1)\pi}{2m+1}\right) \end{vmatrix}. \end{aligned}$$

Now, since

$$\cos(kx) = 2^{k-1} \cos^k x + \sum_{i < k} \alpha_{k,i} \cos(ix), \quad k \geq 2,$$

for appropriate coefficients  $\alpha_{k,i}$ , it follows immediately that the last determinant is the same as the following Vandermonde determinant:

$$(-1)^m 4 \cdot 6 \cdots 2^{m-2} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \cos\left(\frac{2\pi}{2m+1}\right) & \cos\left(\frac{4\pi}{2m+1}\right) & \cdots & \cos\left(\frac{2m\pi}{2m+1}\right) \\ \cdots & \cdots & \cdots & \cdots \\ \cos^{m-1}\left(\frac{2\pi}{2m+1}\right) & \cos^{m-1}\left(\frac{4\pi}{2m+1}\right) & \cdots & \cos^{m-1}\left(\frac{2m\pi}{2m+1}\right) \end{vmatrix},$$

and hence, we obtain the formula

$$\det(\mathbf{Cos}_{2m+1}) = (-1)^{m-1} 4 \cdot 6 \cdots 2^{m-2} \prod_{i < j} \left( \cos\left(\frac{2\pi j}{2m+1}\right) - \cos\left(\frac{2\pi i}{2m+1}\right) \right).$$

Since cosine is decreasing on  $(0, \pi)$  it is easy to calculate the sign of this determinant. We compute its absolute value by squaring the matrix and taking square roots. Indeed, it is straightforward to compute the matrix  $\mathbf{Cos}_{2m+1}^2$  by expressing the resulting sums of products of cosines as the real parts of geometric series of complex exponentials. We omit the details. The result is

$$\begin{pmatrix} \frac{m}{2} - \frac{1}{4} & -1/2 & \cdots & -1/2 \\ -1/2 & \frac{m}{2} - \frac{1}{4} & \cdots & -1/2 \\ \cdots & \cdots & \cdots & \cdots \\ -1/2 & -1/2 & \cdots & \frac{m}{2} - \frac{1}{4} \end{pmatrix}$$

and the determinant of this matrix is easily found to be

$$\det(\mathbf{Cos}_{2m+1}^2) = \frac{1}{2} \left( \frac{m}{2} + \frac{1}{4} \right)^{m-1}.$$

Putting everything together one finally obtains the first formulas in the statement of the theorem.

Finally, it is easy to see that the inverse of the matrix  $\mathbf{Cos}_{2m+1}^2$  is

$$\mathbf{Cos}_{2m+1}^{-2} = \frac{8}{2m+1} \begin{pmatrix} \frac{3}{2} & 1 & \cdots & 1 \\ 1 & \frac{3}{2} & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & \frac{3}{2} \end{pmatrix}.$$

If we ignore the constant and multiply this last matrix by  $\mathbf{Cos}_{2m+1}$  then, taking into account that the column sums in this matrix are all equal to  $-\frac{1}{2}$ , it is clear that the  $(i, j)$ th entry in the product is  $-\frac{1}{2} + \frac{1}{2} \cos(\frac{2\pi ij}{2m+1})$ . Hence,

$$\mathbf{Cos}_{2m+1}^{-1} = \mathbf{Cos}_{2m+1}^{-2} \cdot \mathbf{Cos}_{2m+1} = \frac{8}{2m+1} \left( -\frac{1}{2} \mathbf{U} + \frac{1}{2} \mathbf{Cos}_{2m+1} \right). \quad \square$$

The explicit formulas in Theorem 3.3 allow us to solve the system (3.7) for any  $m$ . We shall briefly sketch the result for odd  $m$ . The even case is similar, but the expressions that appear are longer. The main point here, in any case, is not the resulting explicit formulas for symmetric combinations of  $Z$  series, which may be obtained by several other methods, but rather those for  $M$  series (see Remark 3 below).

So, when  $m$  is odd, Theorem 3.3 implies that the solutions of the system of linear equations (3.7) are

$$y_r = \frac{4}{m} \sum_{i=1}^{\omega} \left( -1 + \cos \left( \frac{2\pi ir}{m} \right) \right) \left( B_{2k} \left( \frac{i}{m} \right) - \frac{B_{2k}}{m^{2k}} \right)$$

for each  $r = 1, \dots, \omega$ . This expression may be simplified considerably by using the following result.

**Proposition 3.4.** *Let  $m \geq 2$  and  $k \in \mathbb{N}$ . Then*

$$\sum_{i=1}^{\omega} B_{2k} \left( \frac{i}{m} \right) = \begin{cases} \frac{B_{2k}}{2} (m^{1-2k} - 1), & \text{if } m \text{ is odd,} \\ \frac{B_{2k}}{2} (m^{1-2k} + 2^{1-2k} - 2), & \text{if } m \text{ is even.} \end{cases}$$

*Proof.* This follows from the *multiplication formula*

$$B_n(mx) = m^{n-1} \sum_{j=0}^{m-1} B_n(x + j/m)$$

(see, for instance, [1, formula 23.1.10, p. 804]), which is easily proven by using the generating function and the cyclotomic equation.  $\square$

We then obtain the following explicit expression:

**Proposition 3.5.** *Let  $m$  be an odd integer equal to or greater than 3 and  $k \in \mathbb{N}$ . Then*

$$\sum_{n \equiv \pm r \pmod{m}} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} (2\pi)^{2k}}{2(2k)!} \left( \frac{2}{m} B_{2k} + \frac{4}{m} \sum_{i=1}^{(m-1)/2} \cos\left(\frac{2\pi ir}{m}\right) B_{2k}\left(\frac{i}{m}\right) \right)$$

for each  $r = 1, \dots, (m-1)/2$ .

As explained above, a similar formula would arise for even  $m$ , but we omit it.

*Remark 3.* We have not found the formulas in Theorem 3.3 in the literature. On the other hand, Proposition 3.5 can be obtained directly from (2.2) by a straightforward argument consisting of inverting the discrete Fourier transform of  $m$ -periodic even sequences (see [3]). The method described in [3] allows one to sum periodic Dirichlet series in general and does not require separate arguments according to the parity of  $m$ . However, the approach in [3] does not reveal the regularity of (3.7) and the explicit formula for the inverse of the cosine matrix. In addition, since  $\mu(n)$  is not periodic, the method does not provide elementary expressions for the  $x_r$  starting from (2.4), i.e. for the series

$$\sum_{n \equiv \pm r \pmod{m}} \frac{\mu(n)}{n^{2k}}.$$

The approach we give here shows that these sums may be obtained in a similar manner, that is, by proving the regularity of (3.6), using a similar argument to that which we have given for

$$\sum_{n \equiv \pm r \pmod{m}} \frac{1}{n^{2k}},$$

except one must replace the matrix  $\mathbf{Cos}_m$  with  $\mathbf{B}_{2k,m}$ .

To this end, we note that by the results to be proved independently in the next section regarding the asymptotic behavior of the Bernoulli polynomials, we have

$$\lim_{k \rightarrow \infty} \left( \frac{(-1)^{k-1} (2\pi)^{2k}}{2(2k)!} \right)^\omega \det(\mathbf{B}_{2k,m}) = \det(\mathbf{Cos}_m),$$

and hence the regularity of the matrix  $\mathbf{Cos}_m$  implies that of  $\mathbf{B}_{2k,m}$ , at least for  $k$  sufficiently large (this is the small caveat we mentioned). We conjecture that this is true for all  $k \in \mathbb{N}$ , although a direct approach along the lines of that used for the cosine matrices does not seem straightforward. For instance, in [11], the rank of the matrix of fractional parts  $(\{ij/m\})_{i,j=1}^m$  is found to depend on the number of divisors of  $m$ . Since  $B_1(x) = x - 1/2$ , this is related to the odd exponent case  $k = 1$  of our problem.

To sum up, recalling what  $x_j$  is, we may state the following.

**Proposition 3.6.** *Let  $m, k \in \mathbb{N}$ ,  $m \geq 3$ ,  $k \gg 0$ . The value of*

$$\sum_{n \equiv \pm r \pmod{m}} \frac{\mu(n)}{n^{2k}}, \quad r = 1, \dots, \lfloor m/2 \rfloor,$$

is  $\pi^{-2k}$  times a rational linear combination of the values of  $\cos(2\pi x)$  at the rational arguments  $x = j/m$ ,  $j = 0, 1, \dots, \lfloor m/2 \rfloor$ .

Thus, for example, we have

$$\sum_{n \equiv \pm 1 \pmod{9}} \frac{\mu(n)}{n^2} = \frac{9}{\pi^2} \left( \frac{1}{4} + \cos \frac{\pi}{9} - \frac{1}{2} \cos \frac{4\pi}{9} \right).$$

3.4. *Some remarks on the odd power case:  $2k + 1$*

Since  $B_{2k+1}(1-x) = -B_{2k+1}(x)$ , in the odd power case we need to consider differences instead of sums. Let  $m \geq 3$  and define

$$y'_i = D(k)^{-1} (M_m(2k+1, i) - M_m(2k+1, m-i)), \quad i = 1, \dots, \omega,$$

where  $\omega = \lfloor m/2 \rfloor$  if  $m$  is odd (just as for the even power case of  $2k$ ) and  $\omega = \lfloor m/2 \rfloor - 1$  if  $m$  is even (one less equation than for  $2k$ ); in short

$$\omega = \lfloor (m-1)/2 \rfloor.$$

We have two systems of linear equations analogous to (3.7) and (3.6), with matrices

$$\mathbf{B}_{2k+1}(m) = \left( B_{2k+1} \left( \left\{ \frac{ij}{m} \right\} \right) \right)_{i,j=1}^{\omega}, \quad \mathbf{Sin}_m = \left( \sin \left( 2\pi \frac{ij}{m} \right) \right)_{i,j=1}^{\omega}.$$

This case is simpler because the square of the sine matrix is easily seen to be diagonal.

**Proposition 3.7.** *For  $m \geq 3$ ,*

$$\mathbf{Sin}_m^2 = \frac{m}{4} \mathbf{I}_{\omega},$$

where  $\mathbf{I}_{\omega}$  is the identity matrix of order  $\omega$ .

Reasoning in the same way as in the even power case, we arrive at the following.

**Proposition 3.8.** *Let  $m, k \in \mathbb{N}$ ,  $m \geq 3$ . Then*

$$\sum_{n \equiv r \pmod{m}} \frac{1}{n^{2k+1}} - \sum_{n \equiv -r \pmod{m}} \frac{1}{n^{2k+1}} = \frac{4}{m} \frac{(-1)^{k-1} (2\pi)^{2k+1}}{2(2k+1)!} \sum_{i=1}^{\omega} \sin \left( 2\pi \frac{ir}{m} \right) B_{2k+1} \left( \frac{i}{m} \right),$$

for each  $r = 1, \dots, \omega$ .

**Proposition 3.9.** *Let  $m, k \in \mathbb{N}$ ,  $m \geq 3$ ,  $k \gg 0$ . The value of*

$$\sum_{n \equiv r \pmod{m}} \frac{\mu(n)}{n^{2k+1}} - \sum_{n \equiv -r \pmod{m}} \frac{\mu(n)}{n^{2k+1}}$$

is  $\pi^{-(2k+1)}$  times a rational linear combination of the values  $\sin(2\pi x)$  at the rational arguments  $x = j/m$ ,  $0 \leq j \leq m-1$ .

#### 4. Asymptotic formulas for the Bernoulli and Euler polynomials

Let us recall that we require knowing the asymptotic behavior of the Bernoulli and Euler polynomials in order to complete the results of the previous section, by showing that the matrices we defined there in terms of the values at rational numbers of a given Bernoulli polynomial  $B_k$ , are invertible for  $k \gg 0$ . This asymptotic behavior is well-known (see [14]):

$$\lim_{k \rightarrow \infty} \frac{(-1)^{k-1} (2\pi)^{2k}}{2(2k)!} B_{2k}(x) = \cos(2\pi x), \quad x \in [0, 1], \quad (4.1)$$

$$\lim_{k \rightarrow \infty} \frac{(-1)^{k-1} (2\pi)^{2k+1}}{2(2k+1)!} B_{2k+1}(x) = \sin(2\pi x), \quad x \in [0, 1], \quad (4.2)$$

and

$$\lim_{k \rightarrow \infty} \frac{(-1)^k \pi^{2k}}{4(2k-1)!} E_{2k-1}(x) = \cos(\pi x), \quad x \in [0, 1], \quad (4.3)$$

$$\lim_{k \rightarrow \infty} \frac{(-1)^k \pi^{2k+1}}{4(2k)!} E_{2k}(x) = \sin(\pi x), \quad x \in [0, 1], \quad (4.4)$$

the convergence being uniform on  $[0, 1]$ . Indeed, the result generalizes to  $\mathbb{C}$ , with uniform convergence on compact sets. Restricting ourselves to  $[0, 1]$ , as we have done throughout the paper, we observe that the asymptotic behavior of these polynomial families is an immediate consequence of their Fourier expansions. Moreover, the Fourier series allow one to obtain much more information regarding the degree of approximation. Thus, in this section, we will use the Fourier expansions and elementary estimates to obtain not only the asymptotic behavior, but also explicit bounds for the differences between the polynomials and their limits, as well as for the ratios of successive differences.

To simplify notation here and in the results that follow, we let

$$\tilde{B}_{2k}(x) = \frac{(-1)^{k-1} (2\pi)^{2k}}{2(2k)!} B_{2k}(x), \quad \tilde{B}_{2k+1}(x) = \frac{(-1)^{k-1} (2\pi)^{2k+1}}{2(2k+1)!} B_{2k+1}(x),$$

and

$$\tilde{E}_{2k-1}(x) = \frac{(-1)^k \pi^{2k}}{4(2k-1)!} E_{2k-1}(x), \quad \tilde{E}_{2k}(x) = \frac{(-1)^k \pi^{2k+1}}{4(2k)!} E_{2k}(x).$$

**Proposition 4.1.** *The Bernoulli polynomials satisfy*

$$|\tilde{B}_{2k}(x) - \cos(2\pi x)| < \frac{2k+1}{2k-1} \cdot \frac{1}{2^{2k}}, \quad x \in [0, 1], \quad k \geq 1, \quad (4.5)$$

$$|\tilde{B}_{2k+1}(x) - \sin(2\pi x)| < \frac{k+1}{k} \cdot \frac{1}{2^{2k+1}}, \quad x \in [0, 1], \quad k \geq 1. \quad (4.6)$$

*Proof.* For  $m \geq 2$  and  $s \in \mathbb{R}$ ,  $s > 1$ , consider the tail of the zeta series,  $Z_m(s) = \sum_{n=m}^{\infty} \frac{1}{n^s}$ . We have the elementary estimate

$$Z_m(s) = \sum_{n=m}^{\infty} \frac{1}{n^s} < \int_{m-1}^{\infty} \frac{dx}{x^s} = \frac{1}{(s-1)(m-1)^{s-1}}.$$

This estimate can be improved by feeding it back into itself:

$$\sum_{n=m}^{\infty} \frac{1}{n^s} = \frac{1}{m^s} + \sum_{n=m+1}^{\infty} \frac{1}{n^s} < \frac{1}{m^s} + \frac{1}{(s-1)m^{s-1}} = \frac{s-1+m}{s-1} \cdot \frac{1}{m^s}. \quad (4.7)$$

Consider now the even index case  $s = 2k$ . Separating the first term,  $\cos(2\pi x)$ , in the Fourier series (1.3), the remaining terms are bounded in absolute value by the tail  $Z_2(2k)$ , and thus (4.5) follows immediately from (4.7). In the same manner, (4.6) follows from (1.4).  $\square$

**Proposition 4.2.** *The Euler polynomials satisfy*

$$\begin{aligned} |\tilde{E}_{2k-1}(x) - \cos(\pi x)| &< \frac{2k+1}{2k-1} \cdot \frac{1}{2^{2k+1}}, & x \in [0, 1], \quad k \geq 1, \\ |\tilde{E}_{2k}(x) - \sin(\pi x)| &< \frac{k+1}{k} \cdot \frac{1}{2^{2k+2}}, & x \in [0, 1], \quad k \geq 1. \end{aligned}$$

*Proof.* The only difference between the proof of this proposition and that of Proposition 4.1 is that here we need to estimate the tails of the “odd” zeta series  $Z_m^*(s) = \sum_{n=m}^{\infty} \frac{1}{(2n+1)^s}$  for  $m \geq 1$  and  $s > 1$ . A good bound is obtained by simply observing that  $2Z_m^*(s) = \frac{1}{(2m+1)^s} + \frac{1}{(2m+1)^s} + \dots < \frac{1}{(2m)^s} + \frac{1}{(2m+1)^s} + \dots = Z_{2m}(s)$ , and hence, by (4.7),

$$\sum_{n=m}^{\infty} \frac{1}{(2n+1)^s} < \frac{1}{2} \cdot \frac{s-1+2m}{s-1} \cdot \frac{1}{(2m)^s}, \quad m \geq 1, \quad s > 1. \quad (4.8)$$

(This is a slightly better bound than that obtained from the estimate  $Z_m^*(s) < (2m)^{-s} + (2m+2)^{-s} + \dots = 2^{-s} Z_m(s)$ .) The result now follows from (4.8) in the same manner as Proposition 4.1 follows from (4.7).  $\square$

Obviously, the asymptotic formulas (4.1), (4.2), (4.3) and (4.4), as well as the uniform convergence, follow immediately from the previous two propositions.

With the same technique we can also determine the asymptotic rates of decrease of the error at each successive step in the approximation of both the sine and the cosine by means of the Bernoulli and Euler polynomials. Namely, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\tilde{B}_{2k+2}(x) - \cos(2\pi x)}{\tilde{B}_{2k}(x) - \cos(2\pi x)} &= \begin{cases} 1/4 & \text{if } x \in [0, 1] \setminus \{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\}, \\ 1/9 & \text{if } x = \frac{1}{8}, \frac{3}{8}, \frac{5}{8} \text{ or } \frac{7}{8}, \end{cases} \\ \lim_{k \rightarrow \infty} \frac{\tilde{B}_{2k+3}(x) - \sin(2\pi x)}{\tilde{B}_{2k+1}(x) - \sin(2\pi x)} &= \begin{cases} 1/4 & \text{if } x \in (0, 1) \setminus \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}, \\ 1/9 & \text{if } x = \frac{1}{4} \text{ or } \frac{3}{4}, \end{cases} \end{aligned}$$

for the Bernoulli polynomials, and

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\tilde{E}_{2k+1}(x) - \cos(\pi x)}{\tilde{E}_{2k-1}(x) - \cos(\pi x)} &= \begin{cases} 1/9 & \text{if } x \in [0, 1] \setminus \{\frac{1}{6}, \frac{1}{2}, \frac{5}{6}\}, \\ 1/25 & \text{if } x = \frac{1}{6} \text{ or } \frac{5}{6}, \end{cases} \\ \lim_{k \rightarrow \infty} \frac{\tilde{E}_{2k+2}(x) - \sin(\pi x)}{\tilde{E}_{2k}(x) - \sin(\pi x)} &= \begin{cases} 1/9 & \text{if } x \in (0, 1) \setminus \{\frac{1}{3}, \frac{2}{3}\}, \\ 1/25 & \text{if } x = \frac{1}{3} \text{ or } \frac{2}{3}, \end{cases} \end{aligned}$$

for the Euler polynomials. As with the asymptotic behavior, these results are an immediate consequence of the sharper explicit estimates for these quotients given below.



**Theorem 4.3.** For the Bernoulli and Euler polynomials, one has the following estimates.

1. Let  $x \in [0, 1] \setminus \{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\}$ . For  $k \gg 0$ , specifically, when  $\frac{2k+2}{2k-1} \cdot (\frac{2}{3})^{2k} < |\cos(4\pi x)|$ , the quotient  $\frac{\tilde{B}_{2k+2}(x) - \cos(2\pi x)}{\tilde{B}_{2k}(x) - \cos(2\pi x)}$  lies between the two bounds

$$\frac{1}{4} \cdot \frac{1 \mp |\sec(4\pi x)| \cdot \frac{2k+4}{2k+1} \left(\frac{2}{3}\right)^{2k+2}}{1 \pm |\sec(4\pi x)| \cdot \frac{2k+2}{2k-1} \left(\frac{2}{3}\right)^{2k}},$$

where the signs are to be taken respectively on top (for the lower bound) and bottom (for the upper bound). If  $x = \frac{1}{8}, \frac{3}{8}, \frac{5}{8}$  or  $\frac{7}{8}$ , then for  $k \geq 2$ , the quotient lies between the two bounds

$$\frac{1}{9} \cdot \frac{1 \mp \sqrt{2} \cdot \frac{2k+5}{2k+1} \left(\frac{3}{4}\right)^{2k+2}}{1 \pm \sqrt{2} \cdot \frac{2k+3}{2k-1} \left(\frac{3}{4}\right)^{2k}}.$$

2. Let  $x \in (0, 1) \setminus \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ . For  $k \gg 0$ , specifically, when  $\frac{2k+3}{2k} \cdot (\frac{2}{3})^{2k+1} < |\sin(4\pi x)|$ , the quotient  $\frac{\tilde{B}_{2k+3}(x) - \sin(2\pi x)}{\tilde{B}_{2k+1}(x) - \sin(2\pi x)}$  lies between the two bounds

$$\frac{1}{4} \cdot \frac{1 \mp |\csc(4\pi x)| \cdot \frac{2k+5}{2k+2} \left(\frac{2}{3}\right)^{2k+3}}{1 \pm |\csc(4\pi x)| \cdot \frac{2k+3}{2k} \left(\frac{2}{3}\right)^{2k+1}}.$$

If  $x = \frac{1}{4}$  or  $\frac{3}{4}$ , then for  $k \geq 2$ , the quotient lies between the two bounds

$$\frac{1}{9} \cdot \frac{1 \mp \frac{k+3}{k+1} \left(\frac{3}{4}\right)^{2k+3}}{1 \pm \frac{k+2}{k} \left(\frac{3}{4}\right)^{2k+1}}.$$

3. Let  $x \in [0, 1] \setminus \{\frac{1}{6}, \frac{1}{2}, \frac{5}{6}\}$ . For  $k \gg 0$ , specifically, when  $\frac{1}{2} \cdot \frac{2k+3}{2k-1} \cdot (\frac{3}{4})^{2k} < |\cos(3\pi x)|$ , the quotient  $\frac{\tilde{E}_{2k+1}(x) - \cos(\pi x)}{\tilde{E}_{2k-1}(x) - \cos(\pi x)}$  lies between the two bounds

$$\frac{1}{9} \cdot \frac{1 \mp |\sec(3\pi x)| \cdot \frac{1}{2} \cdot \frac{2k+5}{2k+1} \left(\frac{3}{4}\right)^{2k+2}}{1 \pm |\sec(3\pi x)| \cdot \frac{1}{2} \cdot \frac{2k+3}{2k-1} \left(\frac{3}{4}\right)^{2k}}.$$

If  $x = \frac{1}{6}$  or  $\frac{5}{6}$ , then for  $k \geq 2$ , the quotient lies between the two bounds

$$\frac{1}{25} \cdot \frac{1 \mp \frac{1}{\sqrt{3}} \cdot \frac{2k+7}{2k+1} \left(\frac{5}{6}\right)^{2k+2}}{1 \pm \frac{1}{\sqrt{3}} \cdot \frac{2k+5}{2k-1} \left(\frac{5}{6}\right)^{2k}}.$$

4. Let  $x \in (0, 1) \setminus \{\frac{1}{3}, \frac{2}{3}\}$ . For  $k \gg 0$ , specifically, when  $\frac{1}{2} \cdot \frac{k+2}{k} \cdot (\frac{3}{4})^{2k+1} < |\sin(3\pi x)|$ , the quotient  $\frac{\tilde{E}_{2k+2}(x) - \sin(\pi x)}{\tilde{E}_{2k}(x) - \sin(\pi x)}$  lies between the two bounds

$$\frac{1}{9} \cdot \frac{1 \mp |\csc(3\pi x)| \cdot \frac{1}{2} \cdot \frac{k+3}{k+1} \left(\frac{3}{4}\right)^{2k+3}}{1 \pm |\csc(3\pi x)| \cdot \frac{1}{2} \cdot \frac{k+2}{k} \left(\frac{3}{4}\right)^{2k+1}}.$$

If  $x = \frac{1}{3}$  or  $\frac{2}{3}$ , then for  $k \geq 2$ , the quotient lies between the two bounds

$$\frac{1}{25} \cdot \frac{1 \mp \frac{1}{\sqrt{3}} \cdot \frac{k+4}{k+1} \left(\frac{5}{6}\right)^{2k+3}}{1 \pm \frac{1}{\sqrt{3}} \cdot \frac{k+2}{k} \left(\frac{5}{6}\right)^{2k+1}}.$$

(Note that the remaining excluded values of  $x$  correspond to the cases when the polynomial and all terms in its Fourier series are null.)

*Proof.* Since the techniques are the same in all cases, we will only outline the proof of the first statement, for the even Bernoulli polynomials.

Let  $\Delta_k(x) = \tilde{B}_{2k}(x) - \cos(2\pi x)$ . As with the first asymptotic results, the leading term in the Fourier series dominates the remaining ones. Thus we separate

$$\Delta_k(x) = \tilde{B}_{2k}(x) - \cos(2\pi x) = \frac{\cos(4\pi x)}{2^{2k}} + \sum_{n=3}^{\infty} \frac{\cos(2\pi n x)}{n^{2k}},$$

where the leading term is  $\ell_k(x) = \frac{\cos(4\pi x)}{2^{2k}}$ . By (4.7), the tail is bounded uniformly in  $x$  by  $\epsilon_k = \frac{2k+2}{2k-1} \cdot \frac{1}{3^{2k}}$ .

Thus we have the approximation  $|\Delta_k(x) - \ell_k(x)| < \epsilon_k$ . The condition  $x \in [0, 1] \setminus \{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\}$  means simply that  $\cos(4\pi x) \neq 0$ , or equivalently,  $\ell_k(x) \neq 0$ , and we then verify that, for such a fixed  $x$ , the ‘‘error term’’  $\epsilon_k$  is always eventually smaller than  $\ell_k(x)$  in absolute value. In this particular case,  $\epsilon_k < |\ell_k(x)|$  translates to  $\frac{2k+2}{2k-1} \cdot \left(\frac{2}{3}\right)^{2k} < |\cos(4\pi x)|$ , which clearly holds when  $k \gg 0$ .

This implies that for  $k \gg 0$ ,  $\Delta_k(x)$  and  $\ell_k(x)$  have the same sign, which is the sign of  $\cos(4\pi x)$ . In particular, the successive quotients  $\frac{\Delta_{k+1}(x)}{\Delta_k(x)}$  are positive. Then, by the triangle inequality,  $0 < |\ell_k(x)| - \epsilon_k < |\Delta_k(x)| < |\ell_k(x)| + \epsilon_k$  and hence, for  $k \gg 0$ ,

$$\frac{|\ell_{k+1}(x)| - \epsilon_{k+1}}{|\ell_k(x)| + \epsilon_k} < \left| \frac{\Delta_{k+1}(x)}{\Delta_k(x)} \right| = \frac{\Delta_{k+1}(x)}{\Delta_k(x)} < \frac{|\ell_{k+1}(x)| + \epsilon_{k+1}}{|\ell_k(x)| - \epsilon_k},$$

which, after some algebraic manipulation, yields the bounds given in the statement of the results above.

In the exceptional cases,  $x = \frac{1}{8}, \frac{3}{8}, \frac{5}{8}$  or  $\frac{7}{8}$ , since  $\ell_k(x) = 0$ , we take the next term in the series as leading term, namely  $\ell_k(x) = \frac{\cos(6\pi x)}{3^{2k}}$ . This works because in fact  $|\cos(6\pi x)| = 1/\sqrt{2}$  for all these  $x$ . Changing  $\epsilon_k$  to the estimate (4.7) for the new tail and proceeding as before, gives the set of ‘‘exceptional’’ bounds also stated above.

The other cases are dealt with in the same manner, identifying the corresponding  $\Delta_k, \ell_k, \epsilon_k$ , noting that for the Euler polynomials we use the bound (4.8) instead of (4.7).  $\square$

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## References

- [1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th printing, Dover, New York, 1972. Electronic copy available in <http://www.math.sfu.ca/~cbm/aands/>
- [2] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, 1976.
- [3] E. P. Balanzario, Evaluation of Dirichlet series, *Amer. Math. Monthly* **108** (2001), 969–971.
- [4] M. Benito, L. M. Navas, and J. L. Varona, Möbius inversion formulas for flows of arithmetic semigroups, *J. Number Theory* **128** (2008), 390–412.
- [5] M. Benito, L. M. Navas, and J. L. Varona, Möbius inversion from the point of view of arithmetical semigroup flows, *Proceedings of the “Segundas Jornadas de Teoría de Números”* (Madrid, 2007), *Bibl. Rev. Mat. Iberoamericana*, 2008, pp. 63–81.
- [6] H. Breitenfellner, A unified Möbius inversion formula, *C. R. Math. Rep. Acad. Sci. Canada* **13** (1991), 39–42.
- [7] E. Cesàro, Sur l’inversion de certaines séries, *Ann. Mat. Pura Appl. (2)* **13** (1885), 339–351.
- [8] P. L. Chebyshev, Note sur différentes séries, *J. Math. Pures Appl. (1)* **16** (1851), 337–346.
- [9] Z. Chen, Y. Shen, and J. Ding, The Möbius inversion and Fourier coefficients, *Appl. Math. Comput.* **117** (2001), 161–176.
- [10] Ó. Ciaurri, L. M. Navas, and J. L. Varona, A transform involving Chebyshev polynomials and its inversion formula, *J. Math. Anal. Appl.* **323** (2006), 57–62.
- [11] P. Codec, R. Dvornicich, and U. Zannier, Two problems related to the nonvanishing of  $L(1, \chi)$ , *J. Theor. Nombres Bordeaux* **10** (1998), 49–64.
- [12] F. Costabile, F. Dell’Accio, and M. I. Gualtieri, A new approach to Bernoulli polynomials, *Rend. Mat. Appl. (7)* **26** (2006), 1–12.
- [13] D. Cvijović and J. Klinowski, New formulae for the Bernoulli and Euler polynomials at rational arguments, *Proc. Amer. Math. Soc.* **123** (1995), 1527–1535.
- [14] K. Dilcher, Asymptotic behaviour of Bernoulli, Euler, and generalized Bernoulli polynomials, *J. Approx. Theory* **49** (1987), 321–330.
- [15] H. M. Srivastava, Some formulas for the Bernoulli and Euler polynomials at rational arguments, *Math. Proc. Cambridge Philos. Soc.* **129** (2000), 77–84.