Malaysian Journal of Mathematical Sciences 16(4): 659–671 (2022)
https://doi.org/10.47836/mjms.16.4.01

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

Third Order Root-Finding Methods based on a Generalization of Gander's Result

Sonia Busquier[1], José M. Gutiérrez[2], and Higinio Ramos*[3]

[1] *Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, Spain*
[2] *Departamento de Matemáticas y Computación, Universidad de La Rioja, Logroño, Spain*
[3] *Departamento de Matemática Aplicada y Grupo de Computación Científica, Universidad de Salamanca, Escuela Politécnica Superior, Zamora, Spain*

E-mail:sonia.busquier@upct.es, jmguti@unirioja.es, higra@usal.es
**Corresponding author*

Received: 26 June 2022
Accepted: 27 November 2022

Abstract

In this paper, a generalization of a classical result by Gander concerning the characterization of third order methods, is addressed. New and classical methods are included in the family. In particular, a new construction of the well-known Chebyshev method is presented. Other methods, based on exponential and logarithmic fittings respectively, are rediscovered too. The proposed methods have a local cubic order of convergence and can be competitive with other third-order methods in the literature, as can be seen from the numerical examples presented.

Keywords: Nonlinear equations; root-finding method; exponential fitting approach; logarithmic fitting approach.

1 Introduction

The solution of nonlinear equations is a task that has attracted the effort of numerous researchers for a long time, and still continues. Our goal is to find the roots of an equation of the form

$$f(x) = 0 \tag{1}$$

on a given domain D, where $f(x) : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a sufficiently differentiable function.

The most common method to do that is Newton's method, which approximates successively in the vicinity of an initial point the function $f(x)$ by the tangent line (for an unconventional formulation of Newton's method see [12, 13, 14, 17]). Other methods based on geometric approximations are the well-known Euler's method [1], which considers approximations by parabolas, and the Halley's method [6, 10], also called the method of tangent hyperbolas, or many others that appear in [3]. It is expected that the methods will work better when the characteristics of the function $f(x)$ are taken into account. This is a common approach in the solution of initial-value problems in differential equations, and has given rise to the so-called adapted methods. Following this approach, in this context we are going to develop two methods that approximate the function $f(x)$ adapted by an exponential function or by a logarithmic function, in both cases containing three parameters.

The paper is organized as follows. In Section 2, we present a modification of a classical result given by Gander in 1985 [7] for the construction of third order methods. The local convergence analysis of some of the methods with asymptotic error constants is discussed in Section 3. Also an initial approach concerning the study of the dynamics of the methods has been presented, considering the quadratic polynomial $p(z) = z^2 - 1$ with $z \in \mathbb{C}$. Implementation details together with some numerical examples are considered in Section 4, to show the efficiency of the proposed methods. Finally, some conclusions of the paper are discussed in Section 5.

2 Derivation of a new family of third order iterative methods

In a classical result given by Gander in 1985 [7] third order methods for solving nonlinear equations $f(x) = 0$ are characterized. Actually, it is shown that they are of the form $x_{n+1} = G(x_n)$ for

$$G(x) = x - H(L_f(x))\frac{f(x)}{f'(x)}, \tag{2}$$

where H is a twice differentiable function around $x = 0$ that satisfies the conditions $H(0) = 1$, $H'(0) = 1/2$ and $|H''(0)| < \infty$ and $L_f(x)$ is defined as the quotient

$$L_f(x) = \frac{f(x)f''(x)}{f'(x)^2}, \tag{3}$$

that usually appears in many expressions of high-order iterative methods.

In view of the expressions that will be obtained for the methods in this paper (see for instance (9) and (12)), and taking into account Gander's result, we can formulate a convergence result for methods of the form $x_{n+1} = S(x_n)$ with

$$S(x) = x + T(L_f(x))\frac{f'(x)}{f''(x)}, \tag{4}$$

where T is an appropriate function as stated in the following theorem.

Theorem 2.1. *Let α be a simple zero of the equation $f(x) = 0$ that satisfies $f''(\alpha) \neq 0$. Let us assume that $T : \mathbb{R} \to \mathbb{R}$ is a differentiable enough function around zero that satisfies $T(0) = 0$, $T'(0) = -1$ and $T''(0) = -1$. Then, the iterative method given by $x_{n+1} = S(x_n)$ where S is defined in (4), is convergent to α with at least order three.*

Proof. The result follows as an application of the Schröder-Traub's theorem. Actually, it is a straightforward calculation that for simple roots, $L_f(\alpha) = 0$. Then $S(\alpha) = \alpha$ if $T(0) = 0$. In addition, if we introduce the notation

$$L_{f'}(x) = \frac{f'(x)f'''(x)}{f''(x)^2},$$

it can be deduced the following expression for the first derivative of $S(x)$ defined in (4)

$$S'(x) = 1 + T(L_f(x))\frac{f''(x)^2 - f'(x)f'''(x)}{f''(x)^2} + T'(L_f(x))L_f'(x)\frac{f'(x)}{f''(x)}.$$

Note that

$$\begin{aligned} L_f'(x)\frac{f'(x)}{f''(x)} &= \frac{(f'(x)f''(x) + f(x)f'''(x))f'(x)^2 - 2f(x)f'(x)f''(x)^2}{f'(x)^4}\frac{f'(x)}{f''(x)} \\ &= 1 + \frac{f(x)f'''(x)}{f'(x)f''(x)} - 2L_f(x) = 1 + \frac{f(x)f''(x)}{f'(x)^2}\frac{f'(x)f'''(x)}{f''(x)^2} - 2L_f(x) \\ &= 1 + L_f(x)L_{f'}(x) - 2L_f(x). \end{aligned}$$

Therefore, we have

$$S'(x) = 1 + T(L_f(x))(1 - L_{f'}(x)) + T'(L_f(x))(1 - 2L_f(x) + L_f(x)L_{f'}(x)),$$

and thus $S'(\alpha) = 1 + T'(0) = 0$ by the hypothesis.

To reach cubic convergence, $S''(\alpha)$ must vanish. After some calculations we obtain that

$$S''(\alpha) = T'(0)L_f'(\alpha)(1 - L_{f'}(\alpha)) + T''(0)L_{f'}(\alpha) + T'(0)(-2L_f'(\alpha) + L_f'(\alpha)L_{f'}(\alpha)))$$

$$\frac{f''(\alpha)}{f'(\alpha)}(T''(0) - T'(0)) = 0,$$

by the hypothesis. This completes the proof. □

Remark 2.1. *As an application of this result, we note that iterative methods in the form*

$$x_{n+1} = x_n - \left(L_f(x_n) + \frac{1}{2}L_f(x_n)^2 + \sum_{j\geq 3} a_j L_f(x_n)^j\right)\frac{f'(x_n)}{f''(x_n)} \tag{5}$$

are cubically convergent to simple roots of $f(x) = 0$. The previous series development must be seen in a formal way, assuming that the corresponding convergence conditions are fulfilled. In fact, these methods could be seen as a generalization of the known as C-methods, which contain extra terms to those of the Chebyshev's method (see [3, 4] for instance).

Remark 2.2. *In particular, if $a_j = 0$ for $j \geq 3$ in (5), it results a new construction of the well-known Chebyshev's iterative method*

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}L_f(x_n)\right)\frac{f(x_n)}{f'(x_n)}. \tag{6}$$

This method and some of its properties have been studied by many authors (see [8, 9] for more details).

Remark 2.3. *It should be noted here that the family of methods given in (5) is essentially the same as the one previously given by Gander in (2). In fact, it is given by*

$$\begin{aligned} x_{n+1} &= x_n - \left(L_f(x_n) + \frac{1}{2}L_f(x_n)^2 + \sum_{j\geq 3} a_j L_f(x_n)^j\right)\frac{f'(x_n)}{f''(x_n)} \\ &= x_n - \left(1 + \frac{1}{2}L_f(x_n) + \sum_{j\geq 2} a_{j+1} L_f(x_n)^j\right) L_f(x_n)\frac{f'(x_n)}{f''(x_n)} \\ &= x_n - \left(1 + \frac{1}{2}L_f(x_n) + \sum_{j\geq 2} a_{j+1} L_f(x_n)^j\right)\frac{f(x_n)}{f'(x_n)}. \end{aligned}$$

Note also that this last family of methods belongs to the family of methods with iterative function $G(x) = x - H(L_f(x))f(x)/f'(x)$, given by (2), where

$$H(L_f(x)) = 1 + \frac{1}{2}L_f(x_n) + \sum_{j\geq 2} a_{j+1} L_f(x_n)^j$$

satisfies the conditions $H(0) = 1$, $H'(0) = 1/2$ and $|H''(0)| = 2a_3 < \infty$.

2.1 Two examples: the exponentially-fitted method and the logarithmically-fitted method

In this section two interesting examples will be considered.

An exponentially-fitted method. For solving the problem in (1) let consider the approximation of $f(x)$ by a function $g(x)$ of exponential-type given by

$$g(x) = a + \exp(c + bx)\,,$$

where a, b, c are unknown parameters to be determined. The idea of using interpolating functions of exponential type has been considered, for instance by S. Amat and S. Busquier [2]. In this work, the global convergence of exponential-type iterative methods (as well as logarithmic-type iterative methods) is considered, by following the patterns given in [11] for other well-known iterative methods.

Assuming that $f(x)$ has at least until second-order derivatives, we impose collocation conditions on the function, the first and the second derivatives at the point x_n, that is,

$$g(x_n) = f(x_n)\,, \quad g'(x_n) = f'(x_n)\,, \quad g''(x_n) = f''(x_n),$$

from which we get a system of equations given by

$$\begin{cases} a + \exp(c + bx_n) &= f(x_n), \\ b\exp(c + bx_n) &= f'(x_n), \\ b^2\exp(c + bx_n) &= f''(x_n). \end{cases}$$

After eliminating the unknown c on this system it results in

$$\begin{cases} b(f(x_n) - a) &= f'(x_n), \\ b^2(f(x_n) - a) &= f''(x_n), \end{cases}$$

from which one readily obtains that

$$a = f(x_n) - \frac{(f'(x_n))^2}{f''(x_n)}, \qquad b = \frac{f''(x_n)}{f'(x_n)}. \tag{7}$$

Finally, using the first equation of the system it follows that

$$c = \ln(f(x_n) - a) - bx_n,$$

and substituting the values of a and b in (7) it holds

$$c = \ln\left(\frac{(f'(x_n))^2}{f''(x_n)}\right) - x_n\,\frac{f''(x_n)}{f'(x_n)}. \tag{8}$$

Now, the root α of $f(x) = 0$ is approximated by the root of the equation $g(x) = 0$, that is,

$$\alpha \simeq \frac{\ln(-a) - c}{b}$$

where we see that necessary conditions for the existence of a real root of $g(x)$ are $b \neq 0$ and $a < 0$.

After substituting the values of a, b from (7) and c from (8) in the expression of the approximate root we get the numerical method given by

$$x_{n+1} = x_n + \ln\left(1 - L_f(x_n)\right)\frac{f'(x_n)}{f''(x_n)}, \tag{9}$$

where $L_f(x_n)$ is given by the quotient defined in (3).

A logarithmically-fitted method. Now consider a function $h(x)$ of logarithmic type for approximating the function $f(x)$ given in the nonlinear equation (1),

$$h(x) = a + c\ln(x + b),$$

where again a, b, c are unknown parameters to be determined. This kind of interpolating functions also appears in [2].

It is assumed that $f(x)$ has at least up to second-order derivatives, and imposing the same tangency conditions at x_n as before, results in:

$$h(x_n) = f(x_n), \quad h'(x_n) = f'(x_n), \quad h''(x_n) = f''(x_n).$$

This leads to the system of equations given by

$$\begin{cases} a + c\ln(x_n + b) & = & f(x_n), \\ \dfrac{c}{x_n + b} & = & f'(x_n), \\ \dfrac{-c}{(x_n + b)^2} & = & f''(x_n). \end{cases}$$

Solving this system in the unknowns a, b, c gives

$$b = -x_n - \frac{f'(x_n)}{f''(x_n)}, \qquad c = -\frac{f'(x_n)^2}{f''(x_n)}, \tag{10}$$

$$a = f(x_n) + \frac{f'(x_n)^2}{f''(x_n)} \ln(-f'(x_n)/f''(x_n)). \tag{11}$$

Now, the root α of $f(x) = 0$ is approximated by the root of the equation $h(x) = 0$, that is,

$$\alpha \simeq -b + \exp(-a/c).$$

After substituting the values of a, b, c from (10) and (11) in the expression of the approximate root we get the numerical method given by

$$x_{n+1} = x_n + (1 - \exp(L_f(x_n)))\, \frac{f'(x_n)}{f''(x_n)}. \tag{12}$$

3 Two local convergence results with asymptotic error constants

In [2] a first approach for the convergence of the iterative methods defined in (9) and (12) was considered. Actually, the following results are established:

Theorem 3.1. *Let $f(x) : D \to \mathbb{R}$ be a three-times differentiable function with a simple zero $\alpha \in D$. If $f'(x) \neq 0$ and sign$(f'(x))(\ln|f'(x)|)'' \leq 0$ for each $x \in D$, then the method (9) converges monotonically to α starting from any $x_0 \in D$.*

Theorem 3.2. *Let $f(x) : D \to \mathbb{R}$ be a three-times differentiable function with a simple zero $\alpha \in D$. If $f'(x) \neq 0$ and $f'(x)f'''(x) \leq 2f''(x)^2$ for each $x \in D$, then the method (9) converges monotonically to α starting from any $x_0 \in D$.*

Now some new local convergence theorems for the iterative methods (9) and (12), are presented, which in addition prove their cubic order of convergence. In the statement of these results we consider the usual notations

$$e_n = x_n - \alpha, \tag{13}$$

$$c_j = \frac{f^{(j)}(\alpha)}{j!\,f'(\alpha)}, j \geq 2. \tag{14}$$

Theorem 3.3. *Assume that $f(x) : D \to \mathbb{R}$ is a sufficiently many times differentiable function with a simple zero $\alpha \in D$, with D an open interval, and let x_0 be an initial guess close enough to α. Then, the exponentially-fitted method defined in (9) has third-order of convergence and the error equation is*

$$e_{n+1} = \frac{2c_2^2 - 3c_3}{3}\, e_n^3 + \mathbb{O}(e_n^4)\,.$$

Proof. Considering the iteration function of the method, given by

$$IE(x) = x + \frac{f'(x)}{f''(x)} \ln\left(1 - \frac{f(x)f''(x)}{f'(x)^2}\right),$$

it holds that

$$\begin{cases} IE(\alpha) &= \alpha, \\ IE'(\alpha) &= IE''(\alpha) = 0, \\ IE'''(\alpha) &= \dfrac{f''(\alpha)^2 - f^{(3)}(\alpha)f'(\alpha)}{f'(\alpha)^2}, \end{cases}$$

with the asymptotic error constant given by

$$A_3 = \frac{f''(\alpha)^2 - f^{(3)}(\alpha)f'(\alpha)}{3!f'(\alpha)^2}\,.$$

Now, the application of the Schröder-Traub's theorem (see [15]) results after some calculus in

$$e_{n+1} = \frac{2c_2^2 - 3c_3}{3}\, e_n^3 + \mathbb{O}(e_n^4),$$

and the conclusion of the Theorem follows. □

Theorem 3.4. *Assume that $f(x) : D \to \mathbb{R}$ is a sufficiently many times differentiable function with a simple zero $\alpha \in D$, with D an open interval, and let x_0 be an initial guess close enough to α. Then, the logarithmically-fitted method defined in (12) has third-order of convergence and the error equation is*

$$e_{n+1} = \frac{4c_2^2 - 3c_3}{3}\, e_n^3 + \mathbb{O}(e_n^4)\,.$$

Proof. Let us consider the iteration function

$$IL(x) = x + (1 - \exp(L_f(x)))\, \frac{f'(x)}{f''(x)}.$$

A straightforward calculation gives

$$\begin{cases} IL(\alpha) &= \alpha, \\ IL'(\alpha) &= IL''(\alpha) = 0, \\ IL'''(\alpha) &= \dfrac{2f''(\alpha)^2 - f^{(3)}(\alpha)f'(\alpha)}{f'(\alpha)^2}. \end{cases}$$

So the method (12) is cubically convergent with asymptotic error constant

$$A_3 = \frac{2f''(\alpha)^2 - f^{(3)}(\alpha)f'(\alpha)}{3!f'(\alpha)^2}\,.$$

Now, the application of the Schröder-Traub's theorem (see [15]) results after some calculus in

$$e_{n+1} = \frac{4c_2^2 - 3c_3}{3}\, e_n^3 + \mathbb{O}(e_n^4),$$

and the conclusion of the Theorem follows. □

Remark 3.1. *Note that the asymptotic error constant of the exponentially-fitted method is always smaller than the asymptotic error constant of the logarithmically-fitted method. Nevertheless, we cannot deduce from it than under the same conditions the first method converges faster than the second one, as can be confirmed by looking at the numerical examples.*

3.1 An interesting dynamical example

In the following numerical experiment the basins of attraction of the fixed points of the iteration maps obtained when methods (9) and (12) are applied to the polynomial $p(z) = z^2 - 1$ with $z \in \mathbb{C}$ are plotted. This is just a first approach to the complex dynamics of both methods. Note that the iteration maps obtained by applying methods (9) and (12) to $p(z) = z^2 - 1$ are, respectively

$$F_1(z) = \left(1 + \ln\left(\frac{z^2+1}{2z^2}\right)\right) z$$

and

$$F_2(z) = \left(2 - e^{\frac{1}{2} - \frac{1}{2z^2}}\right) z.$$

The roots of $p(z)$, $z = 1$ and $z = -1$ are fixed points of F_1 and F_2. We select a region in the complex plane and color in black (white) the initial points whose orbits converge to -1 (respectively to 1) with a certain accuracy and under a prefixed number of iterations (see [16] for more details). The case of the exponentially-fitted method, F_1, does not reveal any kind of surprises (points in the right half-plane converge to 1 and points in the left half-plane converge to -1). That is, the method converges to the root that is closer to the initial point. This situation is shared with other methods, such as Newton's or Halley's methods. However the dynamics of the logarithmically-fitted method, F_2, are quite more intricate even in this simple case, as we can see in Figure 1. Note that there are starting points whose orbit converges to the farthest root, a feature that is shared with Chebyshev's method, for instance.

With this example we just want to highlight the need of a deeper dynamical study of the methods developed in this paper for other functions, with a complete analysis of the fixed points and their attracting character, critical values, and so on. For instance, a general study on quadratic polynomials could be done to check if these methods satisfy the so-called "Cayley test" (see [5] for details). To pass the test, in brief means that the considered method behaves as well as Newton's method for quadratic polynomials, that is, the dynamical plane (basin of attraction of the roots) is given by two semi-planes separated by the equidistant line between the two roots. Note that the iteration maps obtained when methods (9) and (12) are applied to polynomials are not rational maps. Therefore their dynamical study could present behaviors that do not appear in the rational case, such as wandering domains.

4 Numerical Examples

In order to test the performance of the proposed methods we have considered the Newton's method, the methods (9) and (12) developed in this work and other well-known third-order meth-

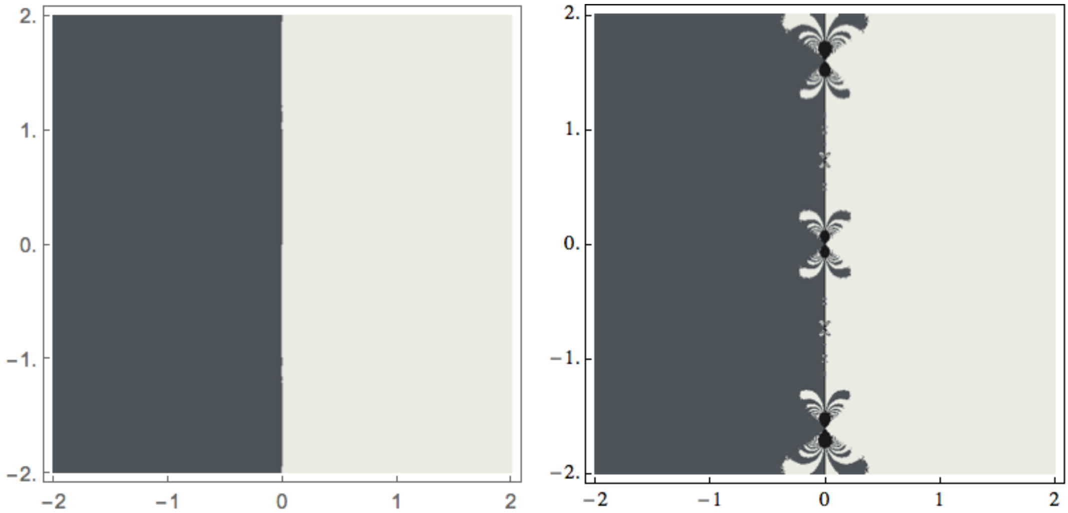

Figure 1: Basins of attraction of the iteration maps $F_1(z)$ and $F_2(z)$, related with the exponentially-fitted and logarithmically-fitted methods applied to $z^2 - 1$.

ods appeared in the literature, which are given as follows:

- Newton's method (NM):
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$
- Euler's method (EM):
$$x_{n+1} = x_n - \left(1 - \sqrt{1 - 2L_f(x_n)}\right) \frac{f'(x_n)}{f''(x_n)}.$$
- Chebyshev's method (CHM):
$$x_{n+1} = x_n - \left(1 + \frac{1}{2} L_f(x_n)\right) \frac{f(x_n)}{f'(x_n)}.$$
- Halley's method (HM):
$$x_{n+1} = x_n - \frac{2}{2 - L_f(x_n)} \frac{f(x_n)}{f'(x_n)}.$$
- The exponentially-fitted method (EFM):
$$x_{n+1} = x_n + \ln\left(1 - L_f(x_n)\right) \frac{f'(x_n)}{f''(x_n)}.$$
- The logarithmically-fitted method (LFM):
$$x_{n+1} = x_n + (1 - \exp(L_f(x_n))) \frac{f'(x_n)}{f''(x_n)}.$$

All the computations were done using the Mathematica system taking 62 digit floating point arithmetics. In Table 1 we show the number of iterations (IT) required to reach a specified accuracy level according to the stopping criterion given by

$$\|x_N - x_{N-1}\| < 10^{-15} \quad \text{and} \quad \|f(x_N)\| < 10^{-15}.$$

The number of maximum iterations before stopping the iterative procedure has been established in 500 (in case of non-convergence before this number of iterations it is indicated by NC). The computational time for each iterative process expressed in milliseconds has been included below the number of iterations. It can be seen that for the exponential and logarithmic-type functions the best performances correspond respectively to the adapted exponential and logarithmic methods, as expected. The following functions have been considered where the approximate zeros α are displayed up to 50th decimal places.

$$f_1(x) = \exp(10x) - 1, \quad \alpha = 0.$$

$$f_2(x) = \exp(x^2 + 7x - 30) - 1, \quad \alpha = -10.$$

$$f_3(x) = 3\exp(x) - \exp(3)x, \quad \alpha = 3.$$

$$f_4(x) = \tfrac{1}{5}\exp(x) + 4\exp(2x) - 10,$$

$$\alpha = \ln\left(\frac{\sqrt{4001}-1}{40}\right) \simeq 0.4423346363699767$$

$$f_5(x) = x^2 + \sin(x) + x, \quad \alpha = 0.$$

$$f_6(x) = \left(x^3 + 1\right)\cos\left(\frac{\pi x}{2}\right) + \sqrt{1 - x^2} - \frac{2}{27}\left(9\sqrt{2} + 7\sqrt{3}\right), \quad \alpha = 1/3.$$

$$f_7(x) = (x-1)^3 - 1, \quad \alpha = 2.$$

$$f_8(x) = \arctan(x), \quad \alpha = 0.$$

$$f_9(x) = \ln(100x) - 1, \quad \alpha = \frac{e}{100} \simeq 0.0271828182845904$$

$$f_{10}(x) = \ln(x^2 + 7x - 30) - 1,$$

$$\alpha = \frac{\sqrt{169+4e}-7}{2} \simeq 3.205839382840827$$

Remark 4.1. *Note that for the equation $f_1(x) = 0$ the exponentially-fitted method is exact, while for the equation $f_9(x) = 0$ the logarithmically-fitted method is exact. This means that only one iteration is necessary to get the exact roots (without taking into account round-off errors). In Table 1 there appear two iterations because the stopping criterion $\|x_N - x_{N-1}\| < 10^{-15}$ must be satisfied.*
We also note that among all the methods considered, only Halley's method and the exponentially-fitted one are capable of obtaining the roots for all the equations.

Table 1: Number of iterations and CPU times needed to get the root with a prescribed tolerance ($*$ means Overflow and $\circ$ means Underflow) for Newton's, Euler's, Chebyshev's, Halley's, exponentially fitted and logarithmically fitted methods.

Function	x_0	NM	EM	CHM	HM	EFM	LFM
$f_1(x)$	-4	$*$	$\circ$	$*$	24	2	$\circ$
		–	–	–	0.75	**0.03**	–
$f_2(x)$	-8	$*$	NC	$*$	15	6	26
		–	–	–	1.41	**0.71**	2.58
$f_3(x)$	12	16	14	11	9	6	10
		0.59	1.16	0.58	0.44	**0.25**	0.32
$f_4(x)$	5	15	13	10	9	5	10
		0.85	1.19	0.66	0.54	**0.26**	0.74
$f_5(x)$	5	8	7	6	6	5	6
		0.30	0.65	0.23	**0.21**	0.33	0.29
$f_6(x)$	0.8	7	6	5	4	5	5
		0.64	2.30	0.82	**0.58**	0.78	0.91
$f_7(x)$	4	8	6	6	6	5	6
		0.11	0.45	0.12	0.11	**0.10**	0.18
$f_8(x)$	4	$*$	10	$*$	5	6	7
		–	0.19	–	**0.11**	0.16	0.13
$f_9(x)$	4	NC	NC	NC	9	8	2
		–	–	–	0.45	0.53	**0.06**
$f_{10}(x)$	6	NC	10	NC	7	8	5
		–	0.81	–	0.63	0.68	**0.37**

5 Conclusions

In this article an exponentially-fitted method and a logarithmically-fitted method for approximating the roots of non-linear equations have been developed. In fact, these methods are a rediscovery of the methods in [2] and they are included in a general family that has been developed modifying a classical result of Gander. It has been shown that the order of the presented methods is three, and some error equations have been provided. We have compared the proposed methods with other well-known methods of same order. From the numerical examples, we conclude that the proposed methods outperform the other methods for each of the types of functions for which they have been designed, and can be comparable for other kind of functions. Thus, the proposed methods may be considered as alternative methods for solving nonlinear equations, particularly if the functions involved present exponential or logarithmic behaviors. In any case, we note that from our main result, many of the third order methods in the literature can be rediscovered.

Acknowledgement The first author thanks the support provided by Programa de Apoyo a la investigación de la Fundación Séneca-Agencia de Ciencia y Tecnología de la Región de Murcia 19374/PI/14 and by MTM2015-64382-P (MINECO/FEDER). The research of the second author was partially supported by the Spanish Ministerio de Economía y Competitividad under grant MTM2014-52016-C2-1-P. The third author thanks the support provided by the Vicerrectorado de Investigación y Transferencia from the University of Salamanca, trough the Grupo de Computación Científica.

Conflicts of Interest The authors declare no conflict of interest.

References

[1] S. Amat, C. Bermúdez, S. Busquier & S. Plaza (2008). On the dynamics of the Euler iterative function. *Appl. Math. Comput.*, *197*(2), 725–732. https://doi.org/10.1016/j.amc.2007.08.086.

[2] S. Amat & S. Busquier (2001). Geometry and convergence of some third-order methods. *Southwest J. Pure Appl. Math.*, *2*, 61–72.

[3] S. Amat, S. Busquier & J. M. Gutiérrez (2003). Geometric constructions of iterative functions to solve nonlinear equations. *J. Comput. Appl. Math.*, *157*(1), 197–205. https://doi.org/10.1016/S0377-0427(03)00420-5.

[4] S. Amat, S. Busquier & J. M. Gutiérrez (2006). An adaptative version of a fourth order iterative method for quadratic equations. *J. Comput. Appl. Math.*, *191*(2), 259–268. https://doi.org/10.1016/j.cam.2005.06.042.

[5] D. K. R. Babajee, A. Cordero & T. J. R. (2016). Study of iterative methods through the Cayley quadratic test. *J. Comput. Appl. Math.*, *2191*, 358–369. https://doi.org/10.1016/j.cam.2014.09.020.

[6] A. Cordero, H. Ramos & T. J. R. (2020). Some variants of Halley's method with memory and their applications for solving several chemical problems. *Journal of Mathematical Chemistry*, *58*(4), 751–774. https://doi.org/10.1007/s10910-020-01108-3.

[7] W. Gander (1985). On Halley's iteration method. *Amer. Math. Monthly*, *92*(2), 131–134.

[8] M. García-Olivo, J. M. Gutiérrez & A. A. Magreñán (2015). A complex dynamical approach of Chebyshev's method. *SeMA Journal*, *71*, 57–68. https://doi.org/10.1007/s40324-015-0046-9.

[9] J. M. Gutiérrez & J. L. Varona (2020). Superattracting extraneous fixed points and n–cycles for Chebyshev's method on cubic polynomials. *Qual. Theory Dyn. Syst.*, *19*(54), 1–23. https://doi.org/10.1007/s12346-020-00390-5.

[10] E. Halley (1809). A new, exact and easy method for finding the roots of any equations generally, without any previous reduction (Latin). *Philos. Trans. Roy. Soc. London (abridged)*, *18*, 640–649.

[11] A. Melman (1997). Geometry and convergence of Euler's and Halley's methods. *SIAM Rev.*, *39*(4), 728–735.

[12] J. F. T. Rabago (2017). Solving higher-order p-adic polynomial equations via Newton-Raphson method. *Malaysian Journal for Mathematical Sciences*, *11*(1), 41–51.

[13] H. Ramos & M. T. T. Monteiro (2017). A new approach based on the Newton's method to solve systems of nonlinear equations. *Journal of Computational and Applied Mathematics*, *318*, 3–13. https://doi.org/10.1016/j.cam.2016.12.019.

[14] H. Ramos & J. Vigo-Aguiar (2015). The application of Newton's method in vector form for solving nonlinear scalar equations where the classical Newton method fails. *Journal of Computational and Applied Mathematics*, *275*, 228–237. https://doi.org/10.1016/j.cam.2014.07.028.

[15] J. F. Traub (1964). *Iterative methods for the solution of equations*. Prentice-Hall, Englewood Cliffs, New Jersey.

[16] J. L. Varona (2002). Graphic and numerical comparison between iterative methods. *Math. Intelligencer*, *24*(1), 37–46. https://doi.org/10.1007/BF03025310.

[17] M. W. Yusuf, L. W. June & M. A. Hassan (2011). Jacobian-free diagonal Newton's method for solving nonlinear systems with singular jacobian. *Malaysian Journal for Mathematical Sciences*, *5*(2), 241–255.